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INDUCTIVE GEOMETRY.

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OR,

AN ANALYSIS OF THE RELATIONS OF FORM AND MAGNITUDE, COMMENCING WITH THE ELEMENTARY IDEAS DERIVED THROUGH THE SENSES, AND PROCEEDING BY A TRAIN OF INDUCTIVE REASONING TO DEVELOPE THE PRESENT STATE OF THE SCIENCE.

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PREFACE.

A BELIEF that considerable alterations were required in the usual mode of arranging the propositions of Geometry, began to prevail with the author from the commencement of the present work: and, occurring whilst the details of the subject were familiar to him, and the inconveniences of the established principles fully apparent, this belief gradually strengthened into a conviction that such partial changes *should* be rendered unnecessary, by an entirely new method of treating the subject.

The author was aware of the cautious and forbearing hand which innovations in works of elementary instruction require; and, previous to recasting a science so widely and justly admired as the Greek Geometry, gave minute attention to the arguments that appeared to favour or oppose the intended measure. The former seemed to him decisive of the course that it would be expedient to pursue; the necessity of the change appearing indeed almost established by the single fact, that, whilst the elements of Geometry have retained the form impressed upon them by the Greeks, the method of reasoning used by the latter has not only been abandoned, in every other branch of mathematics wherein quantity and position are concerned, but, what bears most

directly upon the present question, the change has been attended with important advantages.

The chief result which the author hoped to secure by the proposed innovation, was such an arrangement of the subject as would enable him to dispense with the distinctions hitherto made between the different branches of Geometry; and thus permit him to treat the problems embraced under the heads of—Synthetic Geometry—Analytic Geometry—and the two Trigonometries, as composing one uniform doctrine, the science of Quantity and Position. The path to be pursued in attaining this object was already developed in the modern works on mixed mathematics, and it only remained to apply the principles there laid down, to the single branch of mathematical science wherein the ancient methods were still used. Following this route, a few elements peculiar to the subject were first considered; and when these were shown to admit the relation of addition, and the units most convenient in estimating each had been assumed, the extensive inquiry that has in view the combination of these elements, or, in other words, the whole body of geometrical science, was at once reduced to a known problem of algebra.

That some sacrifice of clearness and brevity would, in a few instances, result from this general method of treating the subject, was, of course, expected; but this consideration was not allowed to have influence, when placed in competition with the clearness and brevity that, it was foreseen, *must* be gained upon the whole, by arranging the different parts of the subject according to their intrinsic relations, and not simply according to the distinct methods of inquiry that might appear convenient in the several cases discussed.

The principles of classification employed by the author, led him to reduce all the propositions of Geometry to the theories of angles about a point, and of closed and open figures; in other words, to the relations of mere direction, and of figures that return into themselves, or which unite distinct points. Pursuing this idea, it became necessary to displace the circle and the sphere from the forced positions wherein they had hitherto stood in works of geometry, and to determine, in the first instance, the relations of rectilinear polygons and solids, not by reference to portions of spheres or circles, but from comparison with some elementary right lined figures, assumed as a type. The selection of the latter was determined by the process of superposition employed in making the comparison: and the author was thus directed to the right angled triangle, and to a solid that he has ventured to call the "rectangular pyramid." Figures that appear *not only* the most convenient for the purpose in view, but so peculiarly adapted to it, that writers on Geometry have constantly employed them in the same manner; without the use of a type, as a principle of investigation, having apparently occurred to them.

Whilst this coincidence will, it is hoped, go far to show that in changing the customary arrangement of the subject, the author has not deviated unnecessarily from the beaten track; he may be permitted also to regard it as a circumstance favourable to the course that has been chosen; since we must certainly view in that light, the fact, that writers whose methods of obtaining their object did not appear to lead them into the same route, have followed it to so great an extent. This unexpected result was found useful in enabling the writer to avoid, what could not be permitted,

any considerable alterations in the signification of terms sanctioned by use, or in the form of known mathematical expressions. And, accordingly, the only change made, in these respects, will be found, on examination, to be merely apparent; it occurs in the definitions of the terms sine, cosine, tangent, &c., which are not the same with the definitions usually given, but agree with them when we have regard to the customary artifice of making radius equal to unity.

It will now be requisite to say a few words upon the use of the term "induction," as applied to a science that has been supposed incapable of that method of reasoning. The mistake frequently committed on this point, has its origin in the vague and erroneous definition usually given of the inductive process; a species of inquiry not confined to the gradual acquisition of general results from a comparison of the phenomena observed in the external world, but extending to the acquirement of knowledge from a comparison of any phenomena, whether mental or physical. An essential difference exists, it is true, between these two classes of facts, but the method of induction will extend to either; and is, indeed, the only process of reasoning whereby it is possible to pass, at will, and in a given direction, beyond the boundaries of our knowledge. Truths, once acquired, may be imparted by other methods; as, by analysis, which developes the consequences of truths already deduced; or by the synthetic process, used in the Greek geometry; and which, agreeing, in this respect, with the method of induction, arrives at its conclusions from a slow comparison of particular facts, but without indicating either the sources whence the latter were obtained, or the principles employed to govern their selection. That neither synthesis nor

analysis can, individually, replace induction as an instrument of inquiry, is evident from what is here said of them, and the omissions noticed in regard to synthesis become more important from the fact, that the premises, in many cases, conduct to a result having apparently such little connection with the intermediate steps, that its appearance at the end of the process, not unfrequently excites a feeling which borders upon surprise.

But if we also view these three methods of inquiry, solely in relation to the exercise they afford the mind, scarcely a doubt will be entertained as to their comparative merits; since it will undoubtedly be looked upon as more useful to acquire habits of research, than to learn a method of demonstration limited to truths already known, and having little connection with the inquiries that must subsequently occupy our attention. We must, it is true, have regard to other considerations, in forming a just estimate of the method proposed, but as, in the great majority of cases, the chief use of mathematical education consists in the discipline it affords the mind, and the new powers of thought which it developes, the author trusts that he shall not be blamed for treating the subject by the process that seemed best adapted to these important ends.

A third innovation has yet to be defended. The author, by requiring that models shall be exhibited of certain of the figures concerning which he speaks, has laid himself open to the objection urged against the method of Pestalozzi; a plan of instruction that has somewhere been described as offering too little exercise to the faculty of conception, but which all allow to give precise ideas concerning the objects taught, and to form habits of careful observation.

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The method here alluded to, and that has obtained much celebrity both in this country and abroad, may be contrasted with the practice of the French mathematicians; who are accustomed, even when treating of the most complicated problems, to dispense altogether with graphical illustration. A course that undoubtedly leaves much uncertainty on the nature of the proposition discussed, but that is otherwise possessed of many advantages; a fact ably shown in the works of its authors, who have demonstrated by an almost unexampled success, the powers of conception and habits of generalization thus acquired. The choice offered, with respect to a plan of instruction, seemed thus to lie between two methods diametrically opposed. But according to the view taken by the writer in regard to this question, the peculiar advantages of both are capable of being combined: every possible aid—by diagrams—by solid and skeleton models—and by reiterated explanation, should, in his opinion, be afforded to the student, whilst the latter is endeavouring to form precise ideas of those elementary forms whereto constant reference is afterwards to be made. This task *once* accomplished, the subsequent combination of these elements may be left to the student's unassisted powers of conception, and will be found to afford him ample opportunities for that species of exercise in which the method of Pestalozzi has been regarded as deficient.

In concluding these remarks, upon the arrangement adopted in the present work, it will be proper to notice the order wherein the book should be read.

This order varies with the extent to which the student has exercised the faculties required in pursuing a train of original inquiry. And as these powers are not usually

matured at the age when students first join our public colleges, it will be right so to proportion the demands upon the faculties of combination and conception, as to keep pace with the gradual development of the powers in question, and the consequent improvement of the mind.

The plan adopted by the author, with regard to the junior mathematical students of the university, is first to read through the preliminary Section of Part I., illustrating every proposition with solid and skeleton models: and thence to pass to the three first Sections of the Second Part. The knowledge so attained, and which presupposes some acquaintance with simple and quadratic equations, will enable the student to master all that is said in Section II., Part III., together with such portions of the first Section of that Part as may be required in pursuing this course.

When the student has thus, in merely passing over the first *elements* of the subject, learned nearly the whole of plane trigonometry, he will be prepared to enter upon the relations of solid figures, and to comprehend what is said concerning the construction of the tables. With these views, he may read two more Sections of Chap. I., Part I., and Sections I., II. and IV. of Chap. II. of the same Part, followed by Section III. and what remains of Section I. Part III.; and reading at the same time with the other Sections here mentioned of this part, such portions of the Sections IV. and V. as can be comprehended by a person who has not proceeded in algebra further than quadratic equations.

The knowledge which the student will now have attained, embraces plane and spherical trigonometry, and the greater part of the propositions contained in most treatises on synthetic geometry; and prior to entering further upon

the subject, it would be proper that he should commence anew: observing, as he now passes over a subject that has become familiar, a more rigid attention to the inductive order; and reading, what may hitherto have been neglected, the Preliminary Reflections and Suggestions.

The order in which the remainder of the treatise is read, is comparatively of little importance; but it may be remarked, that a student whose time is limited, may omit in Section III., Chap. II., Part II., all that follows article 112; and in the Third Part he may omit the Seventh Section.

The author, in closing a work that for the last two years has occupied the brief moments of his leisure, cannot lay down the pen without a few words in explanation of the occasional inadvertences and press errors that it contains. With a view to the benefit of his students, he undertook to reduce, within a given period, the science of quantity and position to a smaller compass, and to put the whole in a form that should render it immediately dependent on the portion of their course that preceded, and in unison with portions that were to follow: the distance at which he resides from the press, and some additional duties, have rendered it necessary for him, in completing this task, to give less than due care to the labour of revisal.

ANALYSIS.

PART I.

THE SUBJECT CONSIDERED IN CONNECTION WITH MATERIAL OBJECTS, AND INSTRUMENTS.

CHAPTER I.

CLASSIFICATION OF THE VARIETIES OF FORM.

PART II.

DETERMINATE ANALYSIS.

PRELIMINARY REFLECTIONS.

Geometry treats of the relations of place, points are the symbols of place, geometry treats then of the relations of points.

Inquiry suggested by these Reflections.

An examination of the most obvious relations of a finite number of points.

CHAPTER I.

FIRST PRINCIPLES OF THE SCIENCE OBTAINED FROM THE RELATIONS OF A FINITE NUMBER OF POINTS.

a point—unit of angles—opposite, or vertical angles are equal—closed solids—solid angles—their unit—geometric analysis conducted by closed figures or solids. *Page* 102.

PRELIMINARY REFLECTIONS TO SECTIONS III. AND IV.

From Sections I. and II., we learn that all the relations of a finite number of points can be obtained from those of closed figures and solids:—such figures and solids can be decomposed into others more simple, and again compounded by putting these simple figures together.

Inquiries suggested by these Reflections.

Can all closed figures be decomposed until their parts are alike:—what simple figure, or type, results from this decomposition:—in what manner are these types to be placed side by side so as to compose any given figure?

Can the same be done for closed solids—in what manner are we to discover the relations of these simple figures, or types of comparison?

SECTION III.—*Relations of the Type to which Closed Figures are compared.*—Relations of three points resumed—symmetry of figures—principle of elementary figures—the triangle which has one right angle assumed as a type of comparison for other triangles—the relations of this type deduced from the principle of superposition—relations of the type—comparison of other triangles with the type performed by superposition—further remarks on the connection between the parts of the type—various principles on which the science may be founded—infinity of space a notion essential to geometry—principle of homogeneity—symmetry of figures. *Page* 119.

SECTION IV. *Relations of the Type to which Closed Solids are compared.*—Inclinations of lines with planes—the triangular pyramid with one solid right angle—the rectangular pyramid—the rectangular pyramid chosen as the type of closed solids—relations of the type—angle which measures the inclination of a line to a plane—particular cases of such angles—measure chosen for the inclination of planes shown to fulfil the necessary conditions—comparison of closed solids with their type—vertical solid angles are equal—comparison of solid angles contained by two planes with those contained by three or four. *Page* 137.

PRELIMINARY REFLECTIONS TO SECTION V.

The type of closed figures, and the type of closed solids, have each had their properties reduced to algebraic equations; but to render these results extensively useful, the process of compounding all figures from their types should also be represented by equations.

Inquiries suggested by these Reflections.

Closed figures and solids being compounded of their respective type, it is required to discover so uniform a method of effecting this composition, that in all cases, the results may be represented by the same equations.

SECTION V.—*General Method of Comparing all Figures with the Type.* —Use of parallel lines and planes as instruments to compare compound figures with the right angled triangle, and compound solids with the rectangular pyramid—theory of parallel straight lines—theory of parallel planes—inclinations of lines that do not meet—general method of comparing linear figures with their type—elementary proposition on which this analysis is founded—property of closed figures which expresses the comparison sought. *Page* 159.

PRELIMINARY REFLECTIONS TO SECTION VI.

Our idea of distance, or *linear quantity,* was obtained by considering the two given points that determined the direction of a line, as its boundaries. But lines themselves are the boundaries of planes, and planes the boundaries of solids.

Inquiry suggested by these Reflections.

By considering, then, lines as the boundaries of planes, should we not obtain an idea of *superficial quantity;* and by considering planes as the boundaries of solids, an idea of *solid quantity*?

These three species of quantity, being alike parts of space; ought they not to have some relation to each other?

Lastly, what modifications will these new ideas require in the analysis of the preceding sections?

SECTION VI.—*Of the Different Species of Quantity which are included in the relations of a Finite Number of Points.*—The three dimensions of space —of space which possesses but two of these dimensions—its unit—of space which possesses the three dimensions—its unit—five units employed in geometry—area of the type of plane figures—solidity of the type of solid figures—the general analysis of figures extended to their relations of area and solidity. *Page* 173.

CHAPTER II.

OF THE ELEMENTS TO WHICH PLACE IS REFERRED.

PRELIMINARY REFLECTIONS.

Hitherto we have considered each problem as distinct, but all geometrical propositions can only be parts of an infinite series of relations connecting all the points in space—these relations resulting from the positions of the points, it follows, that if we could express and tabulate the positions of all points in space, such a table would implicitly contain their relations.

But how is such a table to be formed—for space being uniform and infinite, place is only relative? *Place*, then, *must be rendered absolute* by assuming some fixed point to which all others shall be referred.

But the references to this point will be, first, distance; and secondly, directions according to which the distances are measured.

It will be necessary, then, not only to assign a point whence our measurements shall commence, but also directions according to which they shall be reckoned.

Inquiries suggested by these Reflections.

To how many invariable or *primordial* elements must we refer the position of a point, in order to distinguish it from any other point in space?

Calling the measurements taken with reference to these primordial elements, co-ordinates, how shall we express the relations of points in terms of their co-ordinates?

Section I.—*Various Systems of Primordial Elements.*—Data involved in the position of a point—position of a point merely relative, and determined by relation to primordial elements—relations of a point to the primordial elements expressed by co-ordinates—examination of the cases that occur when the co-ordinates are rectangular—these various methods of expressing place depend on three primordial elements and three co-ordinates—further examination of the cases that occur—oblique co-ordinates—names of the primordial elements and of the co-ordinates—distinction between linear and polar co-ordinates—the parts of an open polygon assign the position of a point—subject of art. 88 continued—method of projections—perspective representations—remarks concerning different cases which occur in the method of projections. *Page* 202.

Section II.—*The Elementary Relations of Points expressed in terms of their Co-ordinates.*—Geometry depends on two elements, the distance between two given points, and the angle formed by two given lines—theorem

for shifting the origin—distance between two points expressed in terms of their linear co-ordinates—angle formed by two given lines expressed in terms of the elements that assign their direction—theorems for transforming the co-ordinates. *Page* 216.

PRELIMINARY REFLECTIONS TO SECTION III.

Applying the formulæ of the last or preceding Sections to the relations of a given number of points, the problem may always be reduced to equations; but varying the positions of the points, not only do different cases of the problem arise, but the equations undergo corresponding variations. Are we then to write the equations of each case, or would it not be possible to obtain a rule for deducing the equations of one case from those of another? And if this is possible, might not the application of that rule change any deductions from the equations of one case into deductions applicable to another?

Inquiries suggested by these Reflections.

Assuming one of the most general cases of a problem as a type of them all—deducing the equations of the type, and thence obtaining algebraic expressions for the properties sought: it is required, first, to trace the alterations which the type undergoes whilst passing into the other cases of the problem: secondly, to trace the corresponding alterations in the equations of the type: thirdly, the alterations in the algebraic expressions of the properties sought: and, lastly, to discover a rule whereby either of the two latter species of variations may be found from the former.

SECTION III.—*Theory of Correlations.*—Problems are resolved by a particular case of the problem taken as a type—the analysis of a geometrical proposition resolves itself into two parts; first, the analysis of a type peculiar to the proposition, secondly, an inquiry into the changes which the type undergoes—the first branch of this double analysis performed by regarding the type as composed of one or more closed figures—method of auxiliary elements—the analysis performed by primordial elements—the second branch of the analysis, or the method of correlations—demonstration of a general rule which connects the changes of the diagram with those of the equations—correlations of angles—angles greater than unity—sequence in which lines and angles are to be estimated—various forms of the theorem which expresses the relations of closed figures—correlation of figures that are used simultaneously—method of avoiding the superfluous equations that would result from the preceding rules. *Page* 227.

PRELIMINARY REFLECTIONS TO SECTION IV.

All closed figures being compared with the right angled triangle, the relations of the latter should be expressed with the utmost possible simplicity:

we have not, hitherto, formed a simple, or even a manageable expression, for the ratio of the sides, in terms of the angles. How is the deficiency to be supplied? The best substitute seems to be a table. A table may be formed by assuming an angle, as small as any that we have occasion to use, and by calculating the ratios in question for every whole multiple of this small angle.

Inquiries suggested by these Reflections.

Relations of angles about a point. Numerical value of the sine of a small angle: simple method of obtaining by consecutive calculations the sines, cosines, &c. of every multiple of this angle.

SECTION IV.—*Relations which the Angles of the Type bear to the Ratios of its Sides.*—Measurement of angles, and notation used to express their quantity—possibility of reducing the sine of any angle to that of a small aliquot part of it—the other ratios may be obtained from the sine—the sine of a particular angle found—calculation of the sine of a very small angle—and thence of all the ratios of any angle whatever—limits of the ratios—their algebraic signs. *Page* 269.

PART III.

ANALYSIS OF PARTICULAR PROBLEMS.

PRELIMINARY REFLECTIONS.

The preceding analysis referring wholly to the relations of points, its further development may naturally be divided into distinct propositions, according as the relations of three, four, or a greater number of points are inquired into.

The relations of two points have been fully developed.

A single point has no relations.

And, yet, if we suppose the angles formed about a point, these relations would include all the relations of direction that could occur in any of the propositions just mentioned. For supposing in space any number of simple directions, and that lines are drawn parallel to them from a given point, the angles formed by these lines will be equal to the angles formed by the directions to which they are parallel.

In treating of such angles, as many distinct divisions can be made, as in treating of the relations of points.

Arrangements suggested by these Reflections.

First Division. Relations of the angles about a point. *Subdivisions.* Dependent on the number of divergent lines.

Second Division. Relations of any number of points in space. *Subdivisions.* Dependent on the number of points.

These general principles of arrangement admit of modification whenever principles of a more partial kind tend to further—the sole object of classification—the ready acquirement and use of knowledge.

Such a subordinate principle arises from the facility with which graphic models can be delineated on plane surfaces; and hence the angles formed by divergent lines that lie in one plane will form the subject of a separate section.

CHAPTER I.

DETAILED ANALYSIS OF THE RELATIONS OF DIRECTION; AND OF THE RELATIONS PECULIAR TO THREE—TO FOUR—AND TO A GREATER NUMBER OF POINTS.

analysis—table of the formulæ used when one solid angle is 90°—apply to the case wherein a plane angle is right—Napier's rules—three parts, in the most general case, determine the remainder—enumeration of data—two plane angles and an opposite solid angle—or the converse—two plane angles and the included solid angle—two solid angles and the interjacent plane angle—another solution of the two preceding cases—three sides—three angles—properties common to every case of three divergent lines—particular relations of three divergent lines—ambiguous cases—appendix. *Page* 324.

Appendix to Section III.—*Examples*. *Page* 355.

Section IV.—*Relations of two Points restricted to a Given Distance and a Given Plane.*—Equation of the circle—a straight line cannot intersect a circle in more than two points—line which is a tangent to a circle—the product of conjugate secants is independent of their direction—the arc of a circle intercepted by two straight lines that diverge from the centre measures their inclination—the sine, cosine, &c. of an angle may be expressed in relations of the arc that measures it—circumference compared with the radius—the inclination of two secants is measured by the difference of the arcs intercepted between them—when the secants intersect in the circumference, their inclination is measured by half the arc intercepted—to find the radius of a circle that shall pass through three given points—to find the radius of a circle that shall be inscribed in a given triangle—differentials of the trigonometrical functions—imaginary formulæ connecting the arc with the trigonometrical functions of it—formulæ of Demoivre—formulæ of Euler—expansions of cos. x^m and sin. x^m. *Page* 367.

Section V.—*Relations of Two Points restricted to a Given Distance.*—Equation of the sphere—the section of a sphere by a plane is a circle—the tangent plane to any point of a spherical surface is at right angles to the radius which passes through that point—if through any point secants are drawn to the sphere, their properties will be the same as those of the secants of the circle—solid angles at the centre of the sphere are measured by the portion of the spherical surface intercepted by them—the solid angle formed by two secant planes is measured by the difference of the spherical surfaces which they intercept—the shortest distance between two points on a sphere is the arc of a great circle intercepted between them—the extremities of the perpendicular drawn from the centre of a sphere to a circle of the latter are every where equally distant from the circle—the angles and sides of a spherical polygon have the same relations as the parts of the solid angle which the polygon subtends at the centre—relations common to all spherical triangles—relations of particular spherical triangles—formulæ for determining the parts of a spherical triangle when one of those parts is 90—polar spherical triangle—formulæ for determining the parts of any spherical triangle—measure of the surface of a spherical triangle—spherical trigonometry includes plane trigonometry as a particular case—equality by symmetry. *Page* 385.

SECTION VI.—*Systems of Primordial Elements that have reference to the Sphere; additional Theorems for transforming co-ordinates.*—The position of a point on the surface of a sphere is referred to the centre, to a great circle, and to a point arbitrarily chosen in the latter—of primary and secondary circles—distance of two points in terms of their spherical co-ordinates—relative directions of points expressed in terms of their spherical co-ordinates—transformation of spherical co-ordinates—transformation of polar systems—equations of transformation used by Euler. *Page* 400.

SECTION VII.—*Relations of any number of Divergent Lines.*—Relations of divergent lines agree with those of spherical polygons—equations relative to the inclinations of divergent lines obtained from the sides of closed figures, by equating with zero the common denominator found in the values of the latter—in a plane, lines equally inclined to other lines form the same angle as the latter—applies also to planes which have a common intersection—perpendiculars to the sides of a closed figure, or to planes that include a solid angle, have for their inclinations the supplements of the internal angles of the latter—sum of the interior angles of a polygon—other relations of divergent lines—data required to determine the relations of divergent lines. *Page* 409.

SECTION VIII.—*Relations of any number of Points.*—Number of distances, of plane angles, of planes, and of solid angles involved in the relations of n points—conditions exist involving merely the directions of the points, and a similar remark applies to their distances—use in the analysis of closed figures of the equations deduced for divergent lines—number and nature of the data that assign the relations of n points—examples in the relations of four points. *Page* 422.

PART IV.

INDETERMINATE ANALYSIS.

PRELIMINARY REFLECTIONS.

The principle that led to the ideas of the circle and the sphere admits of a more extensive application. The hypothesis may be generalized. We may assume more than one point as assigned in space, and innumerable points as connected with them by given relations. Lastly, we may investigate the nature of a surfaco wherein all the latter points are found.

Proceeding in this way we shall be conducted to a peculiar science, teaching to arrange lines and surfaces, not by their apparent forms, but the connection which points they contain have with other points that are given.

Inquiries suggested by these Reflections.

Given any number of primordial elements, to find the points that have assigned relations with them.

CHAPTER I.

OF LINES AND SURFACES.

PRELIMINARY REFLECTIONS TO SECTIONS III. AND IV.

In investigating the relations of a definite number of points, the number is, itself, a character, in terms of which the analysis can be arranged. But as this principle of classification manifestly fails when the number of points is infinite, we have yet to supply that deficiency.

Now the analysis of an infinite number of points proceeding by the equations they give rise to, we may adopt a principle of classification extensively used in algebra, and arrange the several steps of the process by the degrees of the resulting equations.

Inquiry suggested by these Reflections.

To discover the curve, or surface, formed by all those points the relations of which to known elements shall be expressed in an equation of the second degree.

Section III.—*Of Plane Lines of the Second Order.*—Points which lie in a plane are assigned by reference to two primordial elements—of the parabola—of the ellipse—of the hyperbola—connection of the equations discussed in this section—the curves discussed in this section referred to oblique co-ordinates—conjugate diameters—equation of the hyperbola referred to its asymptotes—polar equations of the curves discussed in this section—every equation of the second degree between two variables will be fulfilled by the co-ordinates of one or other of the curves discussed in this section. *Page* 461.

Section IV.—*Of Surfaces of the Second Order.*—Surfaces arranged by the degrees of their equations—equation of the cylinder—sections of the cylinder—equation of the cone—sections of the cone—surfaces of the second degree—ellipsoid—hyperboloid of one sheet—hyperboloid of two sheets—paraboloid. *Page* 489.

PRELIMINARY REFLECTIONS TO SECTIONS V., VI., VII. AND VIII.

The principle of arrangement that assigns the place of a curve by the degree of its equation, applies only to the curves the equations of which are algebraic. But the idea of a curve, or of a mathematical line of any kind, is derived, as explained in Part I., from the intersection of two surfaces ; and thus every principle of arrangement that applies to the latter species of quantity, applies also to curve lines.

Inquiry suggested by these Reflections.

Of lines considered as the intersections of surfaces. Most commodious method of arranging curve surfaces.

Section V.—*Of Lines considered as the Intersections of Surfaces.*—Remarks on the arrangement of lines and surfaces—lines regarded as the intersections of surfaces—of the helix—of the spiral described by the sun—of curves arranged as the loci of points subjected to known motions. *Page* 511.

Section VI.—*Method of arranging Lines and Surfaces by Parameters.*—A parameter, in the arrangement of mathematical quantities, is a variable by means of which we pass from a subdivision to a division—parameters may be measured as co-ordinates—of simple and complex systems of lines—transformation of parameters—dependent parameters are co-ordinates of points wherein the lines or surfaces of the system intersect a line or surface that does not belong to it—two equations between three co-ordinates and a parameter, indicate a line lying in a given surface—two equations between three co-ordinates and two parameters, indicate a system of curves that do not lie in a surface. *Page* 527.

CHAPTER II.

RELATIONS THAT EXIST BETWEEN THE LINES OR THE SURFACES OF ONE SYSTEM, AND THOSE OF ANOTHER.

PRELIMINARY REFLECTIONS.

We discovered, in discussing the properties of the circle and the sphere, certain lines and planes, named tangents and tangent planes, that had remarkable relations with those figures.

Inquiries suggested by these Reflections.

Do all curves and curve surfaces admit of tangents and tangent planes?

division—of multiple points—criterion by which it is discovered whether a given portion of a curve is concave or convex—points of inflexion—points of reflexion—conjugate points—serpentine points—examples. *Page* 600.

Section V.—*Of Curves Tangential and Normal to Systems, and of the Singular Points of Systems.*—Curves tangential to a system of curves—curves normal to a system of curves—singular points of a system of curves—surfaces tangential to a system of surfaces—envelope of a system of surfaces—surfaces normal to a system—singular points and lines of a system. *Page* 612.

PART I.

THE SUBJECT CONSIDERED IN CONNECTION WITH MATERIAL OBJECTS, AND INSTRUMENTS.

CHAPTER I.

CLASSIFICATION OF THE VARIETIES OF FORM.

SECTION I.

MOST OBVIOUS PRINCIPLES OF CLASSIFICATION.

Forms of external objects—relations observed in the forms of objects—classification of the varieties of form—the comparison of figures reduced to the relations of points in space—the relations of points in space reduced to the relations of points in one plane—geometric investigations are performed by the arrangement of elementary figures—models of angles made use of in geometry—compendious arrangement of angles in series—compendious arrangement of some other simple figures—arrangements of more complex figures—delineations of accidental or arbitrary figures—arrangements of certain complex figures that occur in the arts—arrangement of elementary figures—of a principle of arrangement that is applicable to all figures—nature of a geometrical demonstration—a relation among certain lines may necessarily involve a relation among others.

1. The figures of external objects present such exhaustless variety, that to arrange them in general principles seems at first impossible. How little analogy, for example, exists in the figures of the objects now before me—the pencil, the inkstand, the table, the house! On extending my observation further, what

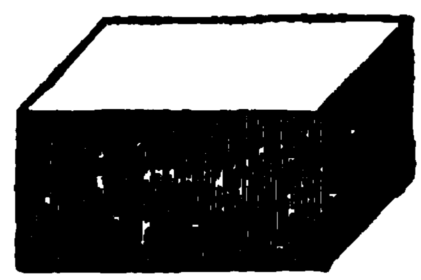
slender relations connect these forms with the bolder and more varied outlines that are seen in the productions of nature!

Figures allowing such abrupt changes would appear related by very complicated laws.

But the fact is otherwise: an attentive comparison of many objects, not only enables us to trace a resemblance between the most dissimilar figures, but demonstrates their points of agreement and gradual transitions to depend on a few obvious principles.

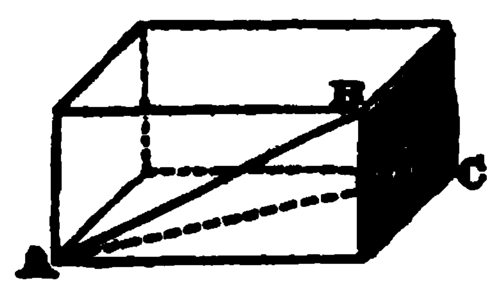
The forms of the table, the inkstand and the house, when closely examined, will be found to have something in common, and to resemble each other by the analogy they bear to an elementary solid that is termed the *right parallelopipedon*.

Fig. 1.



Some of the works of art, such as the blocks of stone or brick or marble used in building, agree so perfectly with this parallelopipedon, that we shall refer to them as sufficiently explaining its nature.

Fig. 2.



A common brick that has its opposite faces equal, and the angles at all its corners the same, is a right parallelopipedon, and a block of some smooth even-grained wood, accurately fashioned into this shape, is the model of the parallelopipedon alluded to in the following examples.

2. Their object, it will be borne in mind, is to show that figures very dissimilar may yet have a connection.

Uniting by straight lines, AB, BC, CA, the corners of the model above mentioned, and cutting the solid in the directions of these lines, or, which answers the pur-

pose, supposing it so cut, there results from the operation a *pyramid*, ABCD, that at the same time differs widely in form from the parallelopipedon, and has obvious relations with it.

The pencil that served for an object of comparison in the beginning of this article will offer us a second illustration of the present subject. As it came to me from the manufacturer, this pretty cedar stick was a little model of a *cylinder*, another elementary form, fig. 3, that frequently occurs in these inquiries. Now if I divide this rod obliquely, and pare, with any sharp tool, the section perfectly even and smooth, the edges of the part so divided present a regular oval, AB, a figure that, seen in the designs of artists, the frames of pictures, or the flower beds of ornamental gardens, rarely suggests the solid to which it bears such intimate relation.

Fig. 3.

The pencil will also supply us with a third example.

A thread evenly wound about this cylinder forms the *helix*, a *spiral curve*, that, ascending by equal advances from the base of the solid towards its opposite extremity, resembles, indeed, the thread of a screw, but appears different from any rectilinear figure that could be traced upon a plane surface.

It has however an immediate dependence on a figure of this kind.

For if a *right angled triangle* (fig. 5), an elementary form that will be understood from the diagram, is cut out of paper, and has its side AB equal to the girt of

the cylinder, and its other side BC equal to the distance between two turns of the thread; this triangle, placed with its acute angle at A, and, wrapped round the cylinder, will follow the spiral line upon the surface of the latter; the side AC continuing to fold itself exactly upon that line, until after one convolution, the angle C falls above the point A, where the triangle was first applied.

Commencing anew at C, and wrapping a second triangle round the cylinder, another convolution is obtained: and continuing this process the whole spiral will be made to coincide with the sides, AC, of these right angled triangles.

Fig. 6.

3. The analogy of figures so different might, of itself, incline us to believe in the existence of elementary relations connecting together the infinite varieties observed in the forms of objects, and permitting the figures exhibited by the latter to be arranged according to fixed principles.

The importance of such a classification can hardly be conceived by those not accustomed to scientific investigations.

Without it the phenomena of the material world, the most extensive portion of nature presented to our view, would be wholly unintelligible; and not only the exact sciences, but—a more important loss—the arts would cease to exist.

How many of the latter depend for example, on a knowledge of *surfaces* and *lines*, particular varieties of form suggested by a comparison of those solid and well

defined objects whose existence is brought home to our minds by the evidence of more than one sense!

The sight and touch incessantly impart information of this nature, and afford us ideas of bodies—not only as substances, resisting the motions of other substances, but as *forms* occupying a definite space.

It is in this light that *geometry* regards all objects.

In the language of that science it is the definite space which is called the *solid*, and not the material object that fills it.

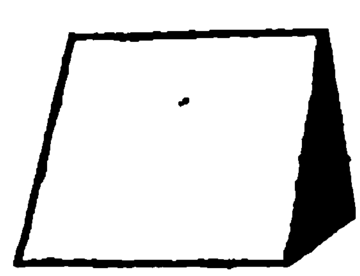
The portion of space exists independently of the object; the latter merely serving, at first, to assist us in conceiving the portion of space, and subsequently, as a symbol to recall its limits.

The boundaries of geometric solids, considered in this way as sensible limits confining definite portions of space, have themselves, in a certain degree, an independent existence—the mind can conceive them when not actually present, can invest them with dimension and colour, and regard them as objects of sensation.

Such boundaries deprived of all physical properties, and looked upon merely as limits of space, are *mathematical surfaces*.

Their arrangement offers obvious distinctions; some solids, as the cylinder, fig. 3, are included by surfaces that present only gradual transitions; whilst the surfaces of others, as the parallelopipedon, fig. 1, or the wedge, fig. 7, exhibit *edges*, or abrupt transitions, dividing them in many cases into distinct surfaces.

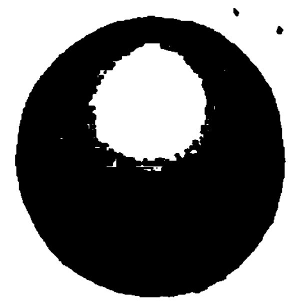
Fig. 7.



The globe, or ball, termed in geometry the *sphere*,

is the most perfect example of a solid bounded by a surface of the former kind: it is in all parts alike, and on this account, and from its simplicity, is much used in comparing other forms.

Fig. 8.



The *cone*, a solid often met with both in natural and artificial objects, is partially bounded by a surface worthy of attention. It forms the type of a class, the "funnel shaped surfaces" of the arts, the "conical surfaces" of geometry, known by these characteristics—they terminate in a point—are in one direction straight—and, finally, capable in that direction of indefinite extension.

Fig. 9.

A sheet of paper loosely folded, but fulfilling these characteristics, offers a familiar example of a conic surface; it will be seen to differ in several respects from the surface fig. 9.

Fig. 10.

The surfaces straight in *one* direction lead us naturally to the surface which is straight in *all*.

We have seen this form, *the plane*, in the boundaries of the solid, fig. 1, where each distinct face constitutes a portion of a surface every where even and straight, and capable of indefinite extension in all directions.

The plane, like the sphere, consists of a single species, but, more restricted than even the surface alluded to, it offers no distinction between the individuals of the class—spheres differ in magnitude, but planes are in all respects alike.

These properties occasion the plane to be more used in geometry than any other surface.

It forms, as we have remarked of the sphere, a type of comparison for other figures, and is the surface on which the pictures of objects are delineated when it is wished to recall their figures.

This circumstance occasions a natural division of figures into two classes, according as they agree with their pictures, that is, have their parts in one plane, or are merely suggested by the pictures, as by signs having a known relation to the thing signified.

The forms included in the first of these classes are termed *plane figures*, and their arrangement has been usually completed in works upon geometry before the figures of the second class were examined.

But however simple this division appears, it is not that followed by nature in imparting to us a knowledge of form.

The figures of bodies that can be handled are the first that attract our attentive observation; whilst pictures, and the division they suggest, are little noticed until the objects themselves have become familiar.

These considerations will explain the course we have taken in treating, successively, the sphere, the cone and the plane, as the most simple cases of surfaces that differ in their capacity to enclose a solid. The plane is obviously the least perfect in this respect, but after an acquaintance with solid bodies has once given rise to the idea of a plane, it requires but a slight power of abstraction to consider the latter as, an object by itself, unconnected with any particular solid.

But the study of tangible objects must not be dismissed until we have obtained from it the idea of a class of figures altogether distinct from surfaces, and having

the same relation to the latter that surfaces have to the solids they inclose.

The edges of the parallelopipedon, fig. 1, and of other solids, or, more generally, the intersections of two surfaces, present examples of such figures, and afford perhaps the most accurate ideas we can attain respecting them.

These boundaries of surfaces are called *lines*.

The lines that form the edges of the parallelopipedon are, evidently, among such figures what planes are among surfaces—they are in all parts alike, and capable of indefinite extension towards either extremity.

They are distinguished as *straight* lines.

A hair, or any other fine thread, stretched by a force at each end; offers an example of a straight line that is not the edge of a solid or the boundary of a surface; and reminds us that other views might be taken of the subject we are investigating.

The thread, examined with attention, will, indeed, be recognized as a cylinder whose breadth and thickness bear a small proportion to the length; but since the two former dimensions may be diminished to any extent, a cylinder, or any other straight rod, when the thickness and breadth are "inconsiderable," may, in practice, be regarded as a straight line.

This subject will again occupy our attention when the different classes of lines arising from the intersections of surfaces have been more fully discussed.

Fig. 11.

The cylinder will present a familiar example of such an intersection; the edge which is found at either extremity, when those extremities are not sloping exhibits *a*

circle, a line of the most frequent use both in science and the arts.

The intersections of a cylinder by a plane that cuts it obliquely, we have already seen, fig. 3, to be a figure of a different kind; but a sphere cut in any direction by a plane has a circle for the line of intersection.

Both the circle and the oval, fig. 4, are plane figures, and both, in the property of enclosing a space, resemble the first of the three chief classes into which surfaces were divided.

We shall not, however, pursue this analogy further, but proceed to relations immediately suggested by the view we have taken of lines.

Points, the boundaries of lines, are relations of this nature; and arise from a process of abstraction similar to that already explained.

The parallelopipedon, for example, presents points at its corners; and these points are regarded without dimensions but merely in the sense implied when the surface was said to be without thickness, and the edges without thickness or breadth.

The points presented by a solid are seen in this way to denote merely the *places* occupied by its corners; and which remain the same when the solid is removed.

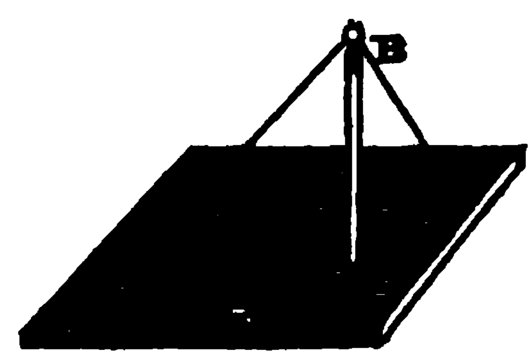
4. This last idea leads to important reflections, and seems to rest the classification of figures on principles different from any we have hitherto considered; the corners of a solid, we argue, determine the directions and length of its edges—the latter determine the faces or planes, that bound the solid; and thus, in all solids that are contained by planes, the whole relations of the figure must be determined by the "*places* of the *corners*;" that is by a definite number of *points in space.*

Let us illustrate this subject by an example: suppose that having before me a model of the parallelopipedon fig. 2, I wish to examine the pyramid ABCD, cut from the solid by the plane ABC, as before described. The actual section, if the solid was of any hard material, would be difficult to execute, and if many different sections were examined it would be previously necessary to obtain an equal number of models.

But regarding the corners of the pyramid as four points in space that determine the relations of the solid, the actual presence of the latter may be supplied by a model of a far more convenient character.

To obtain this model, place the parallelopipedon upon a smooth plane of some even-grained wood, and mark with a pencil the points ACD. Procure a slender rod DB, fig. 12, equal in length to the height of the solid, and having a sharp screw at one extremity. Adjust the rod to the edge DB of the parallelopipedon, and in that situation screw it into the board.

Fig. 12.



With this construction, if the solid is withdrawn the places of the angles A, B, C, D, will still be known.

The place of B will be denoted by the superior extremity of the rod, and those of A, C and D by the pencil marks upon the board.

To restore, therefore, the edges, and study the form of the pyramid, it is only required to unite the points A and B, B and C, A and D, D and C: the two first pair may be united by lines of silk, and the two latter by the pencil.

The facilities which a *linear* model of this kind affords,

both in its construction, in the modifications of that construction, and in the measurement of the several parts, would of themselves entitle it to be regarded as a step in the analysis of form.

The use of the "linear," or "skeleton" model is however far more important when viewed as demonstrating the immediate connection between the forms of objects and the places of certain points in space.

5. The positions of these points we have seen to be readily obtained by lines drawn upon a plane surface, and by slender rods temporarily fastened into it: but a question immediately suggests itself, whether it would not be possible to dispense with these rods, and to determine all the parts of a figure by lines drawn upon a plane surface.

To determine this question it will be necessary to examine separately each face of the model. Let us commence with the face ADB, fig. 13, and 14. A piece of thin board or metal, *m*, accurately fitted into the opening formed by the line AD and the rod DB, will form a measure, or model, of that opening easily transported to any other place. Draw on any smooth surface the line AD, adjust one edge D*a*, of the model *m* to this line, and mark with a pencil the direction of the other edge: remove the model, and produce, with a straight rule, the line D*b* as far as may be necessary. Adjust a straight rule to the line AD, fig. 13, and mark upon it the positions of the points

Fig. 13.

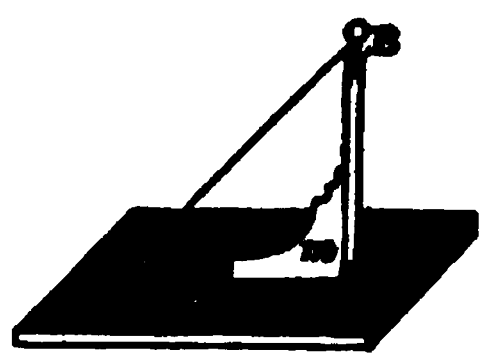
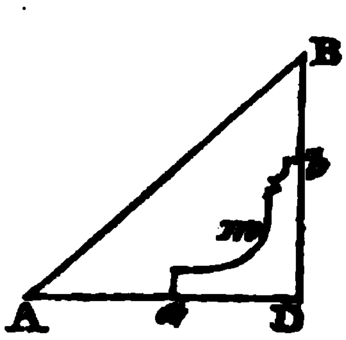


Fig. 14.



A and D. The edge of the rule taken between these points is manifestly a copy, or model, of the line AD, fig. 13, and transporting this copy to fig. 14, adjusting it to the line AD, and, finally, transferring the points from the rule to the paper, or other surface, on which AD is drawn, we obtain there an exact copy of the line of the same name in fig. 13. A similar process applied to DB enables us to transport that line, also, to the surface on which 14 is delineated.

By this process the points of BD are transferred from the linear model to the surface of fig. 14, without deranging their relative positions.

Hence adjusting the edge of the rule to the points A and B, fig. 14, and connecting the latter with the pencil, we obtain a figure, 14, drawn with the pencil upon a plane surface, that is an exact copy of the face ABD of the linear model; and that, placed in contact with this face, could be made to coincide with it line for line.

The process used to obtain a picture of the face in question, fig. 12, would equally apply to BDC; and the pictures of both might, in this way, be traced upon the same surface.

A similar remark applies to the face ADC.

And thus, to obtain on the same surface complete delineations of every part of the solid, it only remains for us to draw there a picture of the face ABC.

But in preparing to effect this operation a reflection will naturally occur that involves the most important consequences.

The fourth face will be seen to result from the three first; and the pictures already completed of these three will enable us to obtain that of the fourth, without refer-

ring for that purpose either to the linear or the solid model.

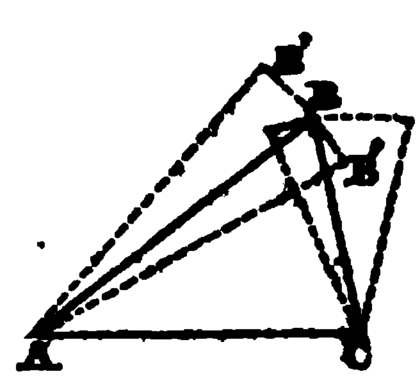
The fourth face is bounded by the lines AB, BC, AC; and either of these will also be found as a boundary of some one of the three faces alluded to.

From the latter we can therefore obtain these lines, without the necessity of referring to the model; and, thus, if a knowledge of the lines bounding the fourth face is sufficient to determine their directions, the delineation of that figure can be performed from the data afforded by the pictures of the three other faces.

The process to which so much importance has been attached is therefore reduced to the solution of the following problem:—"Having exact copies of the straight lines that bound or form a given figure of three sides, it is required to place them in the situations they occupy in that figure."

To solve this problem place the line AC in some convenient part of the plane surface where the pictures already completed were drawn.

Fig. 15.



The positions of A and C will then be completely fixed; but with respect to the line AB our information is much less definite: we know that line to terminate in A, but are altogether in uncertainty in what direction to place it.

In this uncertainty a method of proceeding suggests itself that deserves attention, as an instrument of frequent use in similar cases.

The method consists—first—in "supposing" AB placed, successively, in all positions consistent with the con-

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dition that one extremity shall coincide with A: and—secondly—in observing what consequences are dependent on this supposition.

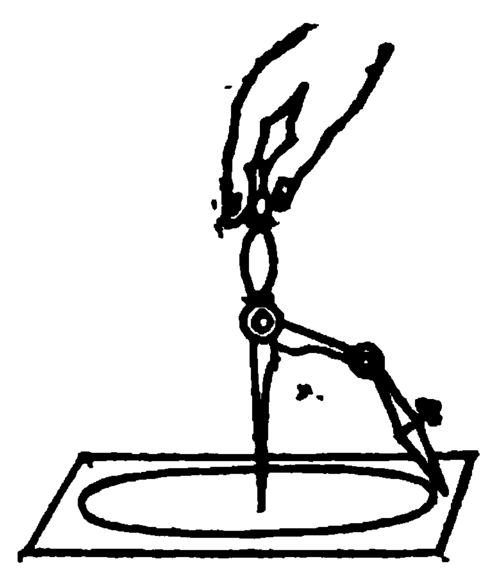
AB, AB′, AB″, fig. 15, may be taken to represent different positions of AB; and it is then immediately seen that B must always lie in some line, B′BB″, which is every where equally distant from A.

A similar result applies to CB; and hence if lines can be drawn, such that one shall have all its points equally distant from A, and the other all its points equally distant from C, the point, B, where these two lines intersect, will determine the position wherein AB and CB are to be drawn.

But the lines here alluded to are readily delineated by means of the mechanical contrivances invented for that purpose.

These contrivances vary with the dimensions of the figure, but whenever the latter does not exceed one or two feet, the *compasses*, fig. 16, are the instruments made use of.

Fig. 16.



The line traced is that already mentioned as the "circle:" and it is easy to see that every point of this curve must be equally distant from the point on which the compasses revolve.

This point is called the *centre*.

The opening of the compasses, or the distance from the centre to the curve, is called the *radius* of the circle.

6. In the problem that led to this digression the points

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A and C were the centres, and the "distances" AB and CB the radii of the circles described; these distances, it will be recollected, were obtained from the pictures of the first and second faces, fig. 12; and as the process we have followed deduces from the latter the "places" of A, B and C, fig. 15, the problem in question is resolved; and the third face determined from the pictures of the first and second.

The lengths of the lines AD, DC and DB, together with the openings ADB, CDB, figs. 12, 13 and 14, were the data, or parts given, and on these the other elements of the figure were shown to depend.

The process used to establish this dependence will be seen, when attentively examined, to consist, merely, of a classification and arrangement of certain figures.

The complex figure, ABCD, fig. 12, was resolved, or "analysed" into four more simple figures, ABD, DBC, ABC and ADC; and then again into forms of a yet more elementary nature, into the straight lines, namely, forming the sides, and into the openings, or *angles*, which those lines inclose.

How completely this resolution is effected may be shown by using frames of wood, or metal, as models of the figures alluded to.

Fig. 17.

Thin laminæ of some similar material may be fitted into the angles.

And it is then apparent that on removing these laminæ, and separating the part of the frame, we obtain a few detached lines and angles that may justly be regarded as the "simple elements" of the complex figure.

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To resolve the latter into such parts is to "analyse it," but as the complex figure may again be constructed by a due arrangement of its elements, the truth of the analysis may be demonstrated by a contrary, or "synthetic" process.

To effect this reconstruction, when all the parts are in our possession, it is only necessary to place the latter according to their known sequence, and to secure them together by any mechanical contrivance adapted to that purpose.

But a far more important problem, we have already remarked, arises when, from certain of the parts or simple elements, we propose to determine the remainder.

The arrangements, we have seen, may then be performed in one plane.

But for this purpose it is necessary that, besides the lines and angles of the given figure, we should possess certain other elementary figures.

These auxiliary elements, in elementary investigations, are circles of different radii, whose office will be recalled to mind by recollecting the method used to determine the point B, fig. 16.

And thus it appears that every step of the preceding investigation may be reduced—first—to an arrangement of figures into certain classes—in the case before us, into straight lines, angles, and circles; and, secondly, to an arrangement of a different kind—to an actual juxtaposition of certain individuals taken from these classes, and placed on a plane surface.

7. Hence, to conduct practically such geometrical investigations, we ought to possess an extensive series of

models representing straight lines, circles and angles of every magnitude.

But as right lines, represented in the preceding example by thin rods, are, strictly speaking, merely the edges of certain solids, our collection of straight lines may be put into a form that is extremely convenient.

Fig. 18.

The edge of a straight rule graduated into minute and equal subdivisions, offers a collection of straight lines of as many distinct lengths as there are divisions.

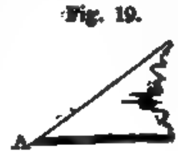
Such a rule, one foot long for example, and divided into hundreds of an inch, affords exact models of twelve hundred straight lines.

And a pair of compasses, since the distance between the points can be adjusted to any of the divisions on the rule, or "scale," will present an equally extensive collection of circles. As compasses cannot however be used when they exceed a certain magnitude, it is not uncommon for artists who have occasion to make geometrical constructions on a large scale, to possess a numerous collection of circular "arcs," cut from laminæ of soft wood.

The series of angles may be presented under a form equally commodious.

But to obtain exact ideas on this subject, it will be proper to consider attentively the elementary figure to which we have given the name of angle.

Fig. 19.



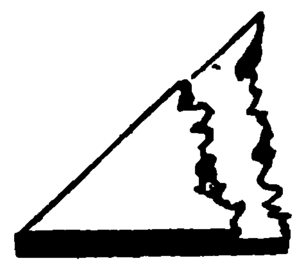
Arranging linear figures according to the classification adopted for surfaces—namely into figures that enclose a space—figures that enclose a space on all but one side, and lastly—figures

that bound space towards one side only: the angle appears to belong to the second of these classes, since on one side it may be continued as far as we please without ariving at any boundary.

The comparison of quantities that in one direction are indefinite, would seem to involve great difficulty. Yet our ideas of the comparison of angles appear precise.

Two angles are said to be equal when, being interposed, they exactly coincide towards those directions where the boundaries are definite. Their coincidence in the remaining direction is not made an object of inquiry.

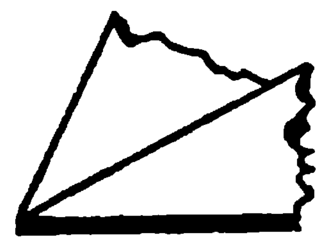
Fig. 20.



But from this measure of their equality it is evident that angles are quantities capable of addition.

Two equal angles if placed side by side will form an angle that is double either of them: and three equal angles if so placed would form a triple angle, &c.

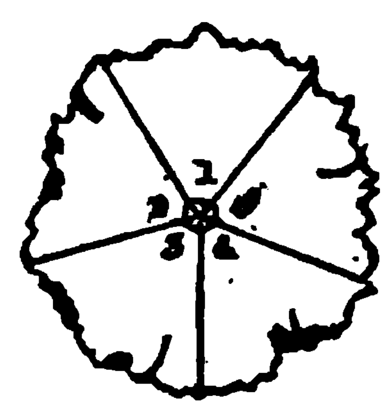
Fig. 21.



Now let us suppose the amount of an angle to be such, that, taking a given number of such angles, five, for example; and placing them together in sequence, the remote side of the fifth should exactly coincide with the near side of the first angle.

An angle of this magnitude would evidently be one-fifth part of the opening about O, or, as all the parts of that opening lie in one plane, the angle in question might be considered as one-fifth part of the *plane space about a point.*

Fig. 22.



Had we, then, models of angles, of all magnitudes,

and in unlimited number, the magnitudes of any two could be compared, by observing what number of each would cover the plane space about a point.

When the angle is not contained an exact number of times in the space about a point, we may arrange side by side as many such angles as can be contained in that space, and afterwards continue the process by means of a second series placed over the first, and resting upon it.

By such an arrangement the magnitude of the angle is found as readily as in the case we have already considered.

Suppose, for example, that twenty-five models of an angle, placed side by side, and, when necessary, in successive layers, were found to cover three times the space about a point—the models forming three layers, or series, about that point, and the remote side of the last model in the upper series falling over the near side of the first model in the lower series.

Since twenty-five times this angle is equal to three times the plane space about a point, the angle itself will be equal to three twenty-fifths of that space; and if it was desired to compare it with the angle before mentioned, an angle contained five times in the plane space about a point, we may do so by considering, that, since one angle is three twenty-fifths, and the other, one-fifth, of the same space: the magnitudes of the angles must be to each other as $\frac{3}{25}$ to $\frac{1}{5}$, or as 3 to 5.

8. It is only in some of the mechanical professions that models of all the angles measured are cut from laminæ of wood or metal.

The carpenter, shipwright, and some other artisans, proceed in this manner, but the geometer, as was before

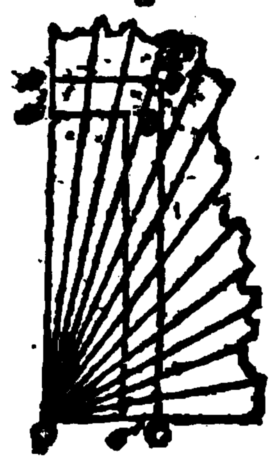
remarked, arranges his series of models in a far more compendious form.

For this purpose he commences with an angle that is an aliquot part of the plane space about a point—and at the same time less than the least angle his calculations require.

Placing side by side a great number of these angles, and drawing upon the plane surface so formed the lines ab, bc, $a'b'$, $b'c'$, fig. 23; he employs an artist to manufacture a scale, fig. 24, of wood, ivory, or metal, that shall present on its edges an exact copy of all the lines found between the boundaries just mentioned.

Fig. 24.

Fig. 23.



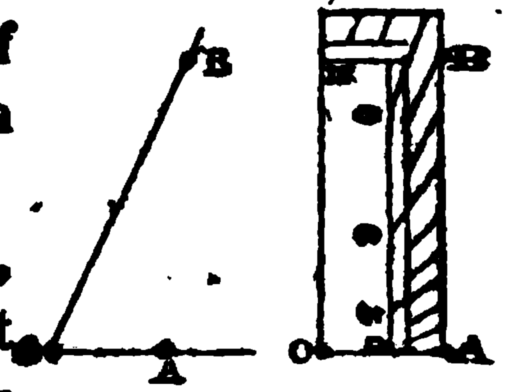
Such a scale presents one of the most commodious forms under which a series of angles can be arranged.

It is named a *protractor*, and is used either for the comparison of angles already delineated, or for tracing upon a plane surface angles of a given magnitude. One example of its application to the latter problem will sufficiently illustrate the use of the instrument.

Let it be required to delineate on some plane surface an angle that shall be one-sixth of the space about a point.

The compound figure 23 is, evidently, the fourth part of that space; and as this figure contains twelve equal angles, eight of them must be equivalent to one-sixth of the space in question.

Fig. 25.



Hence, to trace such an angle, transfer, with a fine pencil, the point O, and the divisions o and 8, from the

protractor to the surface on which the angle is to be drawn, fig. 25.

The point O, and the "places," A, and B, of the divisions o and 8 will still be found on that surface when the instrument is removed; and drawing, with a rule, straight lines through these points, an angle equal to one-sixth of the plane space about a point is delineated on the surface in question.

The form of the protractor, whose use has been illustrated in the preceding example, is not the only one employed, and is indeed much inferior, in respect of accuracy, to an arrangement that records the several angles in the series by divisions not traced, as above described, on the edge of a parallelogram,* but upon the circumference of a circle.

Either instrument will however sufficiently illustrate that classification of figures which has been the object of our research.

And if, bearing this object in mind, we return back on what has been said concerning the scale graduated into equal parts—the protractor—and the compasses, we cannot fail to perceive how materially this classification has been facilitated by compressing into forms so commodious such extensive series of right lines, angles and circles.

9. The earliest geometricians were not acquainted with these instruments in the perfection we now possess them; but the form of the very refined geometry they have left us, sufficiently attests that at an early period instruments had been invented by means of which lines,

* The form of the four sided figure to which this name is given will be sufficiently understood from the diagram.

angles and circles, of definite magnitude, could be delineated at pleasure.

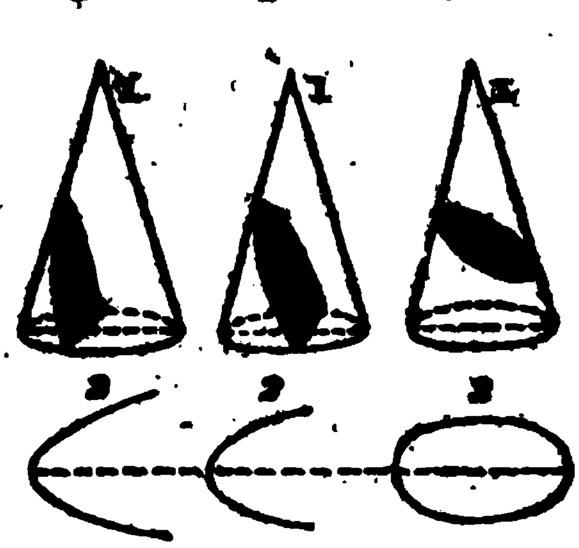
The only figures treated of in the most ancient books of geometry, are such as can be reduced to one or other of these simple forms. But at a later period, they added some classes of figures that cannot be so reduced.

The *sections of the cone* were the chief of these.

The section which a plane makes with a cylinder has already been seen, fig. 3, to form an oval figure, the *ellipse;* a curve of great importance, since it is that regular figure which agrees most nearly with the path described by the planets in their motion round the sun.

The section which a plane makes with a cone is also, in most positions of the former, an ellipse, fig. 26, 1 and 2: but when the plane has a certain direction fig. 27, 1 and 2, the form of the section materially alters, and, from being a closed figure, becomes one of the class that can only partially enclose a space: this figure is called a *parabola.*

Fig. 28. Fig. 27. Fig. 26.



In the ellipse the tranchant plane meets both sides of the cone, and completely divides the superior from the inferior portion of it.

In the position of the plane that causes the section to be a parabola, the plane does not meet one side of the cone, to whatever distance the former is produced.

But a third case, fig. 28, 1 and 2, still remains; it is that wherein the plane, sufficiently produced, meets the side of the cone, also produced, on the exterior of the solid, and beyond the apex.

Such a section is a figure of the same class with the

parabola, but differs from the latter in many essential particulars: it is known as the *hyperbola*.

The ancients formed their series of these figures by means of the solid whence they were derived.

This solid, the cone, might be considered as containing in itself all the figures that could be produced from it by the process we have described; and as constituting, in this way, an extensive magazine of ellipses, parabolas and hyperbolas.

If, for example, to take a familiar illustration of this subject, we had in our possession an unlimited number of ellipses, of all dimensions, and of every form that such a figure admits; we might choose from these models a set of ellipses, and parts of ellipses, so related to each other, that when arranged in sequence on a stand, fig. 29, they should form a cone, fig. 30.

Fig. 29.

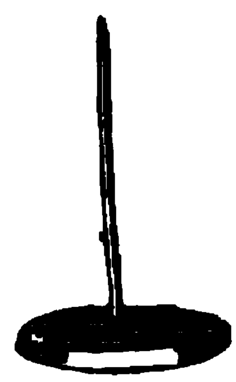
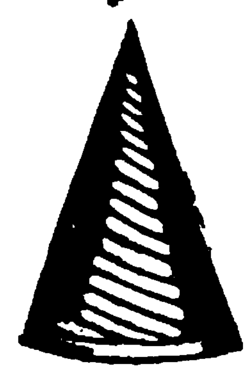


Fig. 30.



The stand is merely a circular base with an upright shaft, and perfectly resembles the file on which it is customary to arrange certain classes of written documents. The ellipses we may suppose cut from thin laminæ of wood, and provided with a perforation, to admit the shaft: their edges are also formed with the obliquity which the surface of the cone requires.

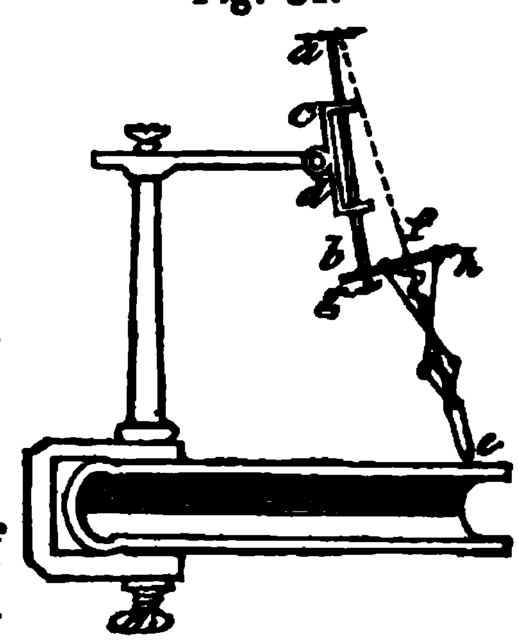
Other cones might be formed of laminæ that were either parabolas or hyperbolas: but such compendious series of these figures are not so perfect as those invented for the series of straight lines, angles and circles: nor is it an easy task to invent any *mechanical* arrangement of ellipses, parabolas and hyperbolas that shall be so perfect.

Instruments that act by continued motion do not apply

so happily to these figures as to the circle: the compasses by which the latter are delineated are machines of the simplest kind, but a complicated mechanism is necessary to the construction of instruments that shall describe conic sections.

A pair of compasses is shown in fig. 31 that will describe either ellipses, parabolas or hyperbolas. Such a machine is introduced in this place not so much from its practical value, as with the theoretic view of presenting a series of conic sections analogous to that used for circles.

Fig. 31.



The cylindric rod *ab* turns freely in the jaws *cd*, that by means of a stiff joint can be placed at any required angle. The pencil *e* is maintained by the rod *ef*, in a direction that passes through, or points towards, *a*: this rod *ef* slides through an orifice in the arm *gh*, and has wound round it a spring that presses the pencil against the surface on which the figure is to be delineated.

The frame work, with joints, attached to the pencil, permits this action of the spring.

Now suppose that by applying the hand at *a*, the rod *cb* is made to turn; it is evident that *ae* must describe the surface of a cone having its apex at *a*. And since the plane against which *e* is pressed intersects this surface obliquely, the pencil will describe a curve that is the intersection of a cone with an oblique plane.

The arm *gh* passes through an opening in the cylindric rod *ab*—by sliding it in this opening the arm can be rendered longer or shorter, and the size of the figure described proportionally increased or diminished.

We can alter, also, at pleasure the inclination of *cd*, and as this inclination determines the obliquity of the plane with respect to the cone, we are enabled to describe any of the conic sections.

10. By other continued motions, or from the intersections of other surfaces, figures different from those hitherto described might be obtained.

But when artists have occasion to delineate such forms, they usually proceed by a method of approximation.

They endeavour to obtain from the problems that give rise to these figures, certain points through which they must necessarily pass: this accomplished, they apply, in succession, to those points a great variety of models, shaped into different figures, and kept for such operations.

Examining these models, they select those among them that will pass through several of the points; and by arranging the selected models together form a curve that passes through all the points required.

This curve is transferred to the surface on which the latter are traced, by a pencil, or other tracing instrument; which is carried along the edge of the moulds.

Fig. 32.

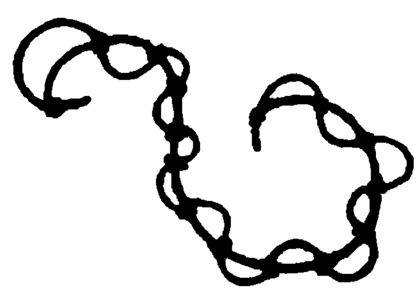
Such an operation is delineated in fig. 32: where two models are so arranged, that, together, they form a curve passing through several given points.

The curve, as it appears after the models are withdrawn, is shown in fig. 33.

Fig. 33.

In this process it is evident that only an approximation to the truth is obtained: the curve, from the conditions of the question, is known to pass through the given points, but as other curves do so likewise, it does not follow that any curve which passes through these points is the figure required by the problem. How widely we may sometimes deviate from the truth by such a supposition is proved in figure 34, where two very different curves are seen to pass through an extensive series of given points.

Fig. 34.



11. Such a process however is often the only one that can be used, and this is especially the case where the figure to be delineated is, in some degree, the result of accident, of caprice, or of causes that lie too deep for investigation.

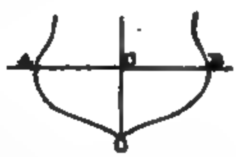
The line formed by a coast, or a river, for example, can only be delineated on the map from a knowledge of numerous points through which it has been found to pass; and a similar remark applies to the figures that result from many of the operations of art.

12. The shipbuilder, for example, wishes to know the line in which the water's surface meets the hull of his vessel.

Ships are composed of planks fastened to upright timbers that nearly resemble in shape and disposition the ribs of the human body. Two such ribs are shown in

figure 35, and exhibit the sections that would be formed if the vessel was cut asunder by a plane that divided the fore from the hind, or as it is technically termed, the *after* part. AB is the line in which the surface of the water is met by this plane, or it is the depth to which the vessel is immersed in that element.

Fig. 35.



The architect, having models of all the ribs before him, measures in each distance the AC, or, in other words, the breadth, at the water's surface, from the ribs to the middle of the vessel.

On some level surface, the floor of a very long room for example, he traces a straight line DE, equal to the whole length of the vessel; and drawing lines across at each of the places where a rib is to be placed, measures on these several lines, respectively, the breadth AC that belongs to the corresponding rib.

Fig. 36.

When this operation has been performed on both sides of DE,* the extremities of these transverse lines will form points through which the intersections sought must necessarily pass.

The remainder of the operation will therefore be reduced to an arrangement of models similar to that, fig. 32, which has already occupied our attention.† (*The*

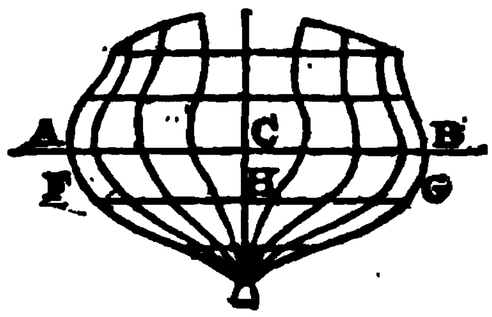
* It is usually performed only on one side.

† On the large scale of which we speak, this arrangement of models is nearly impracticable, and the operation is usually performed by bending an elastic rod until it passes through all the required points. The line so formed is then traced by passing a piece of chalk, prepared for the purpose, along the edge of the bent rod.

arrangement is exhibited, on a reduced scale, in fig. 36.)

From this example it is evident that if we had models of all the ribs, together with an extensive series of models of different figures, we could readily discover, by the process above illustrated, the form of any part of the vessel.

Fig. 37.



The models of the ribs are usually lines drawn on a flat surface, and arranged one within another, as in fig. 37.*

Such a series of figures may therefore be considered as containing within itself the whole form of the vessel, and exhibits a striking classification of the parts of a complex solid.

To discover, for example, the section that could be made by any horizontal plane, we have merely to draw in the series, fig. 37, a line FG, that has the same direction with AB, but lies below it as far as the plane in question lies below the surface of the water. Measuring the breadths of the ribs, (*see the preceding article*) on this line, instead of on AC, and completing the operation performed in the preceding example, fig. 36, the form of the section required is obtained.

13. It would be unnecessary to multiply examples where our only object is to prove, what has already abundantly appeared, that geometry arranges into series, or

* The figure contains the ribs on the fore part only. As both sides are alike, it is usual to delineate only one of them, and to place on the left hand the ribs belonging to the after part, and on the right those belonging to the fore part of the vessel.

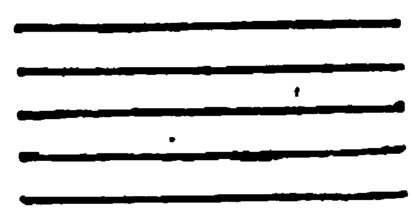
families, those figures that have some natural analogy, or accidental connection of place.

But there are classes of lines which so frequently occur, that, already, they have more than once pressed themselves on our attention, and were with difficulty neglected to allow room for the development of prior ideas.

Parallel lines form one of these classes.

We met with such lines in the edges of the parallelopipedon, fig. 1; and they have since frequently occurred.

Fig. 38.

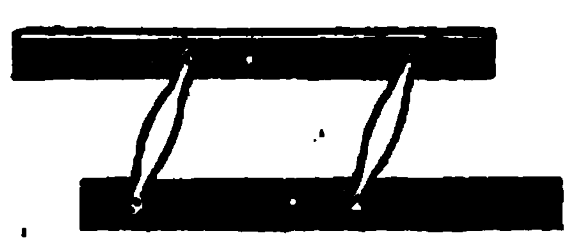


The opposite edges of the parallelopipedon do not meet, however far they may be produced; and it is evident that lines may be drawn on any plane surface that shall possess this property; any straight line, in fact, being drawn upon a plane, innumerable other straight lines may be so traced there, fig. 38, that, when produced, they shall neither meet each other, nor the given line.

Such lines are said to be parallel; and instruments have been invented that serve, when a straight line is delineated on a plane, to develope all the straight lines that lie on that plane, and are parallel to the given line.

In cases where dispatch is more necessary than accuracy, the cylindric rule is used for this purpose. But for operations of a more important character the geometrician employs either the *parallel rule*, fig. 39, or the rule and *square*, fig. 40.

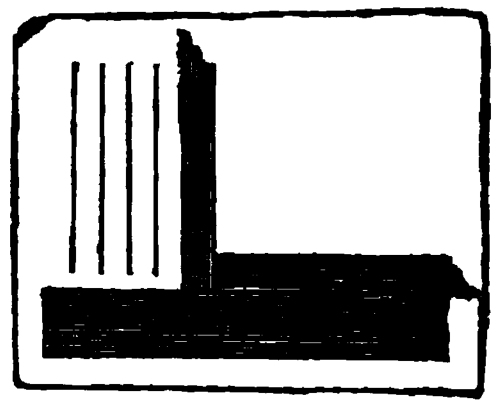
Fig. 39.



The latter is an instrument not yet described, but regarding it, for the present, merely as an invariable angle, we shall readily perceive that when one arm is slid,

fig. 40, along the edge of a straight and immovable rule, the other arm will be carried parallel to itself.

Fig. 40.



Such instruments render all other models of parallel lines unnecessary, and exhibit applications of motion nearly as happy as that which provided us with an infinity of circles.

The "square" mentioned in the preceding article, and so named in the arts, is the model of a particular angle, the *right angle* of geometry.

It is seen in the parallelopipedon, fig. 1, and in the series of angles that are arranged on the protractor, fig. 24. It is the greatest angle that occurs in this last, and is denoted in the figure by the number 12. As the protractor, however, is usually divided into ninety equal parts, or *degrees*, the right angle will be formed on such instruments, opposite to 90, and is hence very frequently called "the angle of ninety degrees."

The relation between the right angle and the plane space about a point will be perceived without difficulty.

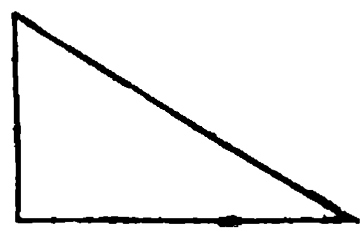
It will be only necessary, for that purpose, to recur to the compound figure 23, in which the protractor originated. The equal angles arranged in that figure were aliquot parts of the plane space about a point, and as the number in the series 23 was one-fourth of that required to complete the space in question, the angle formed by this sum, or the "right angle," must be one-fourth of the space about a point.

Why it should deserve greater attention than the half, the third, or any other aliquot part of this space, is not immediately obvious; but will be gathered from an attentive examination of rectilinear figures.

Classing the latter by the number of their sides, it is chiefly those of three and of four sides that have been subdivided with reference to the right angles they contain.

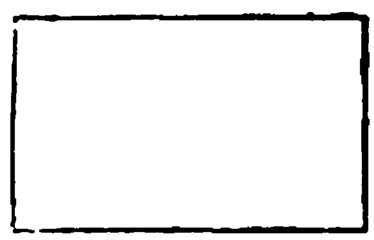
Right angled triangles constitute one of the most simple classes of trilateral figures and will hereafter occupy much of our attention.

Fig. 41.



Among quadrilateral figures the *rectangle* is remarkable for having all its angles right angles.

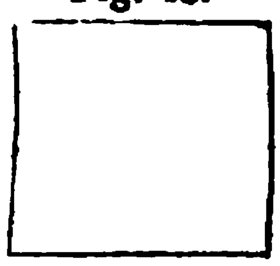
Fig. 42.



This figure we have already remarked in the faces of the parallelopipedon.

When all the sides of the rectangle are equal, it becomes the *geometrical square*; and differs essentially from the figure to which the same appellation is applied in many of the mechanical arts: the latter, as was remarked above, is merely a material model of a right angle.

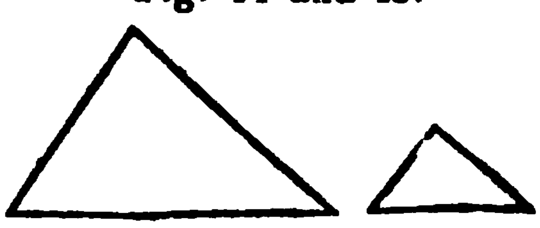
Fig. 43.



14. The last principle of classification that need be mentioned in this place, is, undoubtedly, more important than any that we have hitherto considered. It proceeds entirely on a resemblance inherent to the figures, and not at all on their accidental connections of place.

It is called the principle of *symmetry* or *similarity*, and will be readily understood from a few examples.

Fig. 44 and 45.



According to this principle, the varieties of *form* are distinguished from those of *magnitude*.

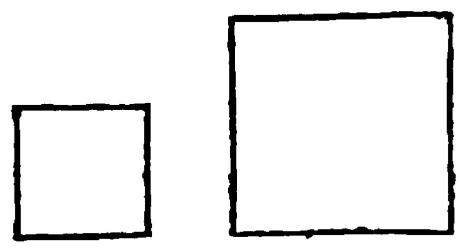
Chap. I. Classification of the varieties of form.

Art. 14. Of a principle of arrangement that is applicable to all figures.

The triangles represented in figures 44 and 45, for example, are evidently of the same form although in magnitude there is considerable difference between them.

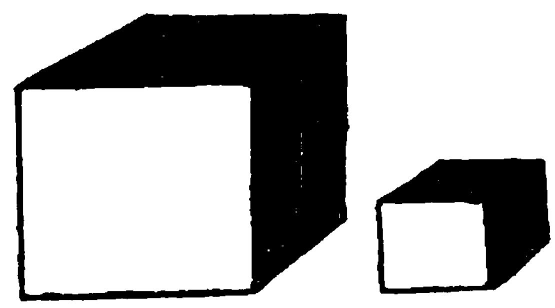
The squares 46 and 47 offer another example of the same kind, and we again meet with an illustration of this principle in the comparison of circles, which, whatever may be the magnitude, are always similar figures.

Fig. 46 and 47.



This want of connection between form and magnitude is equally observable in solids, which admit of an unlimited increase or diminution of bulk, without any corresponding variation in their figure.

Fig. 48 and 49.



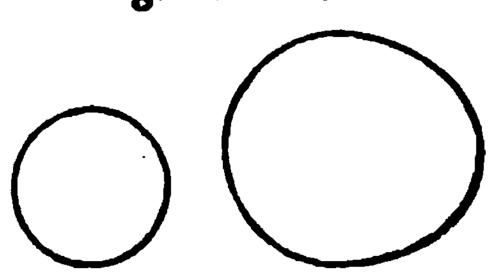
It is evident to how great an extent such a principle must diminish the labour of classing the varieties of form.

In place of using, as we have hitherto done, models equal in magnitude to the objects they represent, this principle of symmetry enables us to substitute models of a more convenient size.

The circle, for example, apparently described by the sun in the heavens, is a figure of precisely the same kind as the curve, figure 16, delineated by a pair of compasses. And whatever relations are discovered in the latter figure must be equally relations of the former.

Thus, to take a familiar example, let it be supposed that we have before us many small diagrams of circles; and let it be further supposed that, on comparing the girths, or *circumferences* of these figures with their radii,

Fig. 50 and 51.



we discover that, in all the circles, the radius is contained a certain number of times in the circumference.

Such a result would constitute a "property" of the circle, or a "relation" between that curve and the straight line which is called the radius.

And if, as we have supposed, this property was proved to be the same in all the diagrams, we might pass at once, by a bold inference, from these minute figures to the orbit which the sun describes, and conclude the property in question to be equally true for that vast circle.

15. But for such an inference to be just it is necessary that our examination of the diagrams should have been conducted according to a method in the highest degree worthy of attention.

If the relation in question was obtained by any mechanical contrivance, or by a comparison that admitted only of approximative results, the conclusion might not be just.

Artists often estimate the circumference of the circle by a process of the former kind: they roll a wheel along a smooth board or sheet of metal, and measure the part rolled over.

Now, although such an operation, if skilfully executed, would give results nearly correct, their accuracy must still be limited by that of the senses.

Were the property we are discussing, then, established by such testimony, we could have no certainty that it extended beyond the figures actually examined.

Deviations too small for the senses to discover in diagrams measured by feet and inches, might rapidly increase with the dimensions of the figure, and, finally, render the supposed property altogether inapplicable.

How often such instances occur may be gathered from the figure of the earth.

The earth, in the small portions of it which the eye at once embraces, seems an extended plane, and measurements made on a surface of a few acres would confirm this result; but a more enlarged inquiry, a measurement that reached beyond the immediate boundary of vision, would not only lead us to modify our first conclusion, but to adopt a figure altogether inconsistent with it.

From these facts, we may conclude that properties extending to figures of every magnitude, are not to be discovered by the methods of inquiry and demonstration made use of in the arts.

But what methods, then, ought to be employed?

Or, since investigation and proof are here synonymous with a comparison of the figures whose arrangement has occupied our attention, we may ask, in other words,

On what principle ought such comparison to be made?

The answer is at once obvious—"The comparison should be *mental.*"

To prove mentally, indeed, that in every circle the circumference contains the same number of times the radius, requires considerable preparation; but this "mental," or "geometrical" proof may be readily illustrated by propositions of a more elementary character; which accomplished, the student will have little difficulty in comprehending the object and nature of GEOMETRY.

"It is the science, he will find, which teaches us to classify the varieties of form, and to discover, by a mental comparison of the figures so arranged, their hidden relations."

The first of these two operations has been already ex-

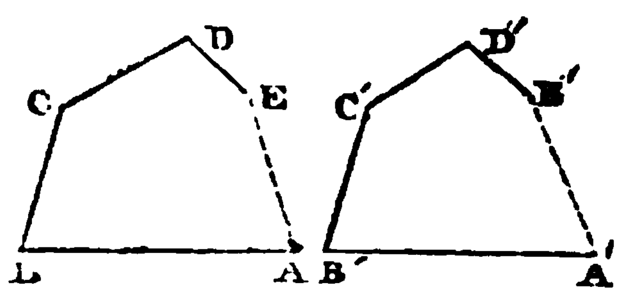
plained, and the second, or the comparison so frequently alluded to in the preceding paragraphs, will be understood from the following simple example:

Let 52 and 53 be two rectilinear plane figures, whose corresponding sides and angles are equal.

It is required, by a strict comparison of these figures, to prove their identity; and to show that if right lines be drawn from the extremity of the first to the extremity of the last side—these lines, which cause the figure to enclose a space, will also be equal.

To prove the first part of the proposition, it is only necessary that we should SUPPOSE the figure A′B′C′D′E′ placed upon the figure ABCDE, or, if the nature of the surface whereon the figures are drawn renders this supposition embarrassing—that we should suppose an exact model, or copy, of fig. 53 carried from that figure and placed on 52.

Fig. 52. Fig. 53.



These two figures may, then, be so adjusted that A′ shall fall upon A, and the line A′B′ take the direction AB.

But when this is effected it involves a necessary CONSEQUENCE: the point B′, necessarily, falls upon B; since if it did not, the sides A′B′ and AB could not be of the same length, a supposition which is contrary to the hypothesis.

This consequence, again, produces another, the points B′ and B coinciding, and the angles A′B′C′ being equal by hypothesis, the sides B′C′ and BC "must" coincide in direction; and, as B′C′ and BC are equal, the point C′ must fall on C.

Continuing this reasoning, it is evident, both that D′ must fall on D and E′ upon E: and the two figures, thus coinciding throughout, are equal in every respect.

16. The second part of the proposition more particularly deserves attention, as asserting the equality of lines that are not assumed to be equal, and which, indeed, are not mentioned in the hypothesis.

Such a result will give rise to important consequences, and offers an example of *a relation among certain lines necessarily involving a relation among others.*

In the case before us it is the equality in the lengths and positions of the sides AB, BC, CD, DE, fig. 52, with the corresponding sides A′B′, B′C′, C′D′ and D′E′, fig. 53, that is asserted to involve an equality between EA and E′A′.

That it does so we have now abundant proof; since, on placing one of the figures, or an exact copy of one of the figures, upon the other, it has been shown that, A′ and E′ can be made to coincide with A and E; and consequently that an exact coincidence can take place between the lines AE and A′E′.

Figures such as these, that is, whose extremities coincide, are called *polygons* or *closed figures;* and act a very conspicuous part in geometrical investigations. Any one side of such figures has been shown to have a necessary relation to the other sides; so that angles and sides taken at pleasure, cannot always be so placed as to form a closed figure.

The reader will by this time, it is hoped, fully comprehend the nature and object of the very important science whose principles we are preparing to investigate.

Sect. I. Most obvious principles of classification.

Art. 16. A relation among certain lines may necessarily involve a relation among others.

He will recollect the successive illustrations of the following results.

First. That by a process of arrangement conducted according to the obvious analogies which figures bear to each other, the complex may be reduced to others more simple, until the forms of an extensive class are made to depend on the places of their more prominent points.

Secondly. That a principle of classification, not less important, distinguishes form from magnitude.

And, *Finally*, That a method of comparison may be employed which, substituting the "mental" for the "actual" superposition of figures, furnishes demonstrations not dependent on the magnitude of the diagram; and applicable therefore to all cases where the form investigated is the same.

SECTION II.

PRINCIPLES CONDUCTING TO A MORE REFINED ANALYSIS.

Example of a geometrical investigation—reflections on the preceding example—branches into which geometry is divided—advantages peculiar to abstract geometry—principles on which geometrical investigations should be conducted—the comparison of figures reduced to the relation of points in space—the magnitude of all the parts of a figure deduced from its form and the magnitude of one part—the investigation of article 17 *made to depend on a single lineal measurement—the investigation of article* 17 *conducted by assuming as known the figure which is the object of inquiry—relations of form and magnitude reduced to the relations of number—recapitulation—elements peculiar to the subject.*

17. The recapitulation that closed the preceding section contains the chief ideas that have resulted from our inquiries, and little as they seem removed from those which a moment's reflection would have suggested, they, nevertheless, constitute facts imperfectly developed in

the earlier works on geometry, and that are capable of leading to the most important consequences.

By their assistance we are enabled to penetrate some of the most hidden secrets of nature; and to answer questions apparently far removed beyond the reach of human inquiry.

How difficult it appears, for example, to estimate the magnitude of the earth!—persons wholly ignorant of geometry are unable to divine the first steps of the inquiry, and look on its success as an ingenious fiction—and yet the solution of this great problem follows immediately from the elementary principles we have developed.

There are two ways in which the form and magnitude of the earth may be determined: one proceeds on a theory respecting the heavenly bodies; but the other is altogether geographical, and measures the earth as we would measure a work of art—a colossal statue, or a ship, or any other object of doubtful figure, and dimensions greatly exceeding our own.

Confining ourselves to this last method, and supposing, for the present, the object to be a work of art; let us consider in what manner we must proceed in order to determine its magnitude and form.

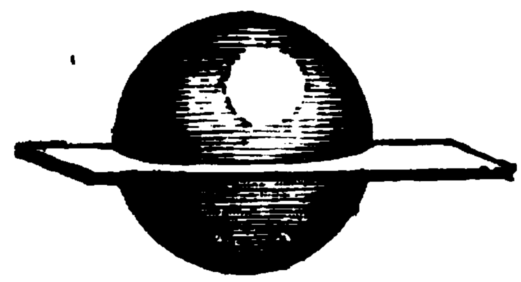
Our knowledge of the forms of objects is best acquired and recorded by means of pictures, and these pictures are most conveniently examined when all the parts of them lie in one plane.

The intersections of planes with the surface investigated are figures of this kind: and when the planes are parallel and placed at equal distances, they afford, we have seen, art. , a series of pictures that express in a very compendious way, the figure of the object.

Hence one of the first steps in the inquiry is to construct, practically, a plane that shall intersect the surface investigated.

If the latter were of small dimensions, the readiest method of attaining this end would be to fashion a thin board, or a plate of metal, into a plane surface: and afterwards, by repeated trials, to shape the edge of this material plane into a figure that could be accurately adjusted to the surface examined. Fig. 54.

Fig. 54.



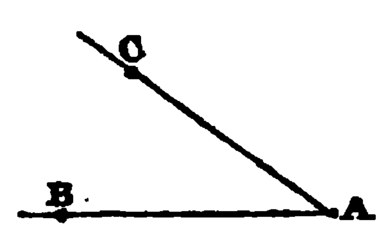
But when the object is of the magnitude which the problem under consideration supposes, such material planes would be altogether unmanageable, and we must have recourse to the resources of geometry for some more refined instrument.

The "mental comparison of figures," the most powerful weapon that geometry employs, will readily furnish a method of this kind. It follows from such a comparison that "*a straight line lies wholly in a plane when it has two points in common with the latter;* and after this abstract truth has been perceived, the practical application of it to the construction of plane surfaces will be readily discovered.

An intermediate step in the process must be demonstrated geometrically.

We see immediately, from the principle in question, that when two straight lines meet they must lie in one plane.

Fig. 55.



For directing the attention exclusively to three points in these lines, the point A, common to both, and the points B and C, taken at pleasure in AB and AC; the ideas we

form respecting planes sufficiently demonstrate that a material model of such a surface could be made to pass through A and B; and it is equally obvious that, when coincident with these points, the plane could be turned upon them as centres.

But if this motion of the material plane is continued, the latter must at some instant pass through C.

The points A, B and C, therefore, lie in one plane.

But the line AB, *theorem, page* 44, is in the same plane with the points A and B, and the line AC is in the same plane with A and C.

Hence we conclude, as was to be shown, that AB and AC are in one plane.

The facts used in this proposition are not obtained from the senses, but are mental perceptions of a class too elementary to admit of further analysis.

Such truisms are called in geometry *axioms*, and are the elements to which all geometrical demonstrations must be reduced.

The reduction, if developed in every proposition, would render geometrical investigations of inconvenient length, but mathematicians avoid this difficulty, by arranging their demonstrations in a series so constructed, that any proposition is dependent on propositions that preceded it: these, again, depending on others of an earlier rank, until, by thus tracing the series backward, we ultimately arrive at propositions established on the evidence of axioms alone.

This "logical arrangement" is not only an extensive principle of investigation, it is a principle that cannot be neglected, and that must in a great measure govern every geometrical classification of the varieties of form.

We shall give an example of its application to the inquiry before us.

That inquiry—we recollect—has for its ultimate object to discover the figure and magnitude of the earth.

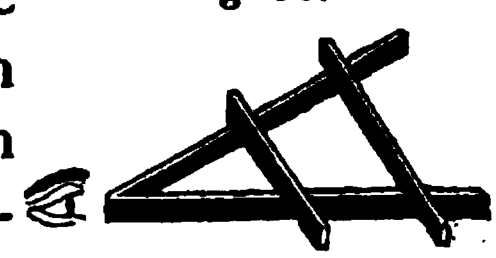
It was introduced as an example of a geometrical investigation, and in prosecuting it we have been led to examine the most practicable methods of constructing plane surfaces.

We have demonstrated that two straight lines which meet, lie in one plane: and in immediate sequence with this proposition we will now arrange a second, demonstrating, that, "if two straight lines which meet in a point, are each met by a third straight line, the three lines will lie in one plane."

By the preceding proposition all the points in the two first lines are in one plane.

But as the points in which the third line meets the first and second, are points in those lines, they lie in the same plane with them. And the plane passing through the first and second lines, has therefore two points in common with the line remaining.

Fig. 56.



But, according to an axiom already quoted, straight lines that have two points in common with a plane, lie wholly in it.

The third line, therefore, lies wholly in the plane that passes through the first and second line, or, in other words, the three lines, as was to be shown, are in one plane.

The practical construction in view is greatly facilitated by this proposition, which enables us to trace every part of a plane by means of an apparatus that is easily arranged and transported.

The model of three straight lines constitutes the whole of this apparatus—fig. 56.

To arrange points in a straight line by this process is

called, in many of the mechanical arts, "to look them out of winding"—that is to judge by the eye when the points are brought into a line that no where "winds," or deflects from a uniform direction. In other of the arts the process is, more simply, termed "looking points into a right line," or—an appellation that we shall hereafter employ—"bringing points into the line of vision."

With the assistance of this optical principle, one of the base lines may be dispensed with.

A number of straight fillets placed upon a single base line may be looked into one plane by directing the eye along their lower edges, and observing when the latter exactly cover or hide each other.

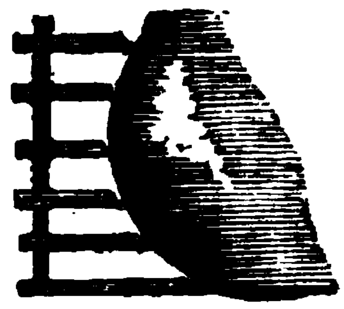
The line of vision is in this case brought to coincide with a point in each fillet, and if the eye is placed at a point, fig. 56, in the base, the line of vision will be a a line meeting the base line and each of the fillets: the base, the fillets, and the line of vision will, therefore, by the last proposition, be in one plane.

This arrangement possesses many advantages over the former, and is indeed so convenient that we may regard as completed the first part of the inquiry proposed in the commencement of this article; and proceed to determine the intersection of the plane, constructed as above, with the surface that it is proposed to investigate.

When, as in the preceding part of this article, a thin board, or any other material lamina, was the plane employed, its edge, fig. 54, could be fashioned into the form of the intersection sought: but in the case we are now considering this method of proceeding is no longer applicable; and we must seek to determine the intersection by assigning the places of a sufficient number of its points, art. 12.

With this view, a base is erected, and straight fillets, arranged at convenient distances upon it, are looked into one plane.

Fig. 57.



The extremity of the fillet which is next to the surface examined terminates in a point.

By the assistance of a square the fillets are placed at right angles to the base.

And in this position, and whilst these straight rods are all in one plane, they are moved towards the object, until the pointed extremity of each is in contact with its surface.

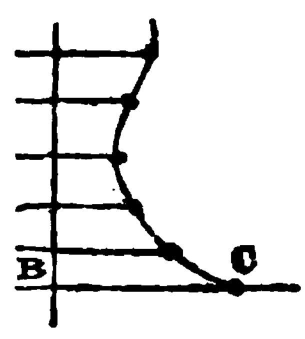
The intersection of the fillets with the base line is then marked upon the former, and the operation, as far as it has in view to obtain an accurate record of this particular section, is complete.

Similar operations are performed on other sections, and when a sufficient number of the latter have been measured, we are in possession of an accurate record of the whole surface.

The process of tracing these interections upon a plane surface, or, in other words, the delineation of their pictures, is a subsequent operation, and one that may be performed at any time.

When a convenient period for that purpose has arrived, a straight line is drawn on some smooth surface, the floor of a large room for example; and the base formerly used having been adjusted to this line, a transfer is made to the latter of those divisions on the base that record the positions of the straight fillets.

Fig. 58.



The line drawn on the plane surface we may consider

as the representation of the base, and the divisions, or points, transferred to it, as the places of the straight fillets.

From each of these divisions lines are drawn perpendicular to the representation of the base, fig. 58; and the fillets having been adjusted to them, the division on each fillet that records its intersection with the base is made to coincide with the representation of the latter.

The whole of these material lines are thus brought into the relative positions which they occupied when the section to be delineated was examined.

Those extremities of the fillets that were then in contact with the object, will now, therefore, have the same relative position.

And hence, the extremities in question, if transferred to the plane surface, will be so many points in the picture to be delineated.

And as we have seen in art. 12, how to describe a line that shall pass through any number of given points; it is only necessary to employ the process there mentioned in order to obtain an exact copy of the intersection sought.

If the body was very irregular, a ship, for example, many intersections would be required.

The example in art. 12 will sufficiently explain the most convenient arrangement they admit of: we see from that example that the investigation of the object should have been conducted by the assistance of planes parallel to each other,* and placed at small and nearly equal distances.

* Parallel planes are those which do not meet however far produced; they will be understood from what has already been said about parallel lines.

But whatever arrangement we adopt, the method of proceeding is the same, and the process used to obtain the representation of one intersection is to be repeated with each.

The forms, indeed, of many bodies, both natural and artificial, are too irregular to permit an exact and convenient measurement by sections, however well chosen or numerous; but, even when this is the case, the same principle of investigation may be employed, and merely requires a more convenient mode of application.

The principle alluded to consisted, we recollect, in measuring the position of a point in the object by its distance from a fixed base line.

The fillet, or material rod, used in measuring this distance, was applied at a known angle with the base, and the place where it met with the latter, or the length of the base, reckoned from some fixed point, was also ascertained.

But the fillet, and that portion of the base which is measured, form, together, two sides of an *open figure* (articles 15 and 16, figures 52 and 53), whose extremities are, the fixed point on the base, and the point sought in the object.

In fig. 58 we observe many of these "open figures" arranged in the same plane; O a b, for example, is an open figure, commencing at the fixed point O, and terminating at b, a point in the object.

This arrangement of many open figures in one plane, the method of application used in the preceding case, fails, we have observed, when the form of the object examined is very irregular; but other modes of arrangement may then be adopted, that shall still render the investigation by open figures available.

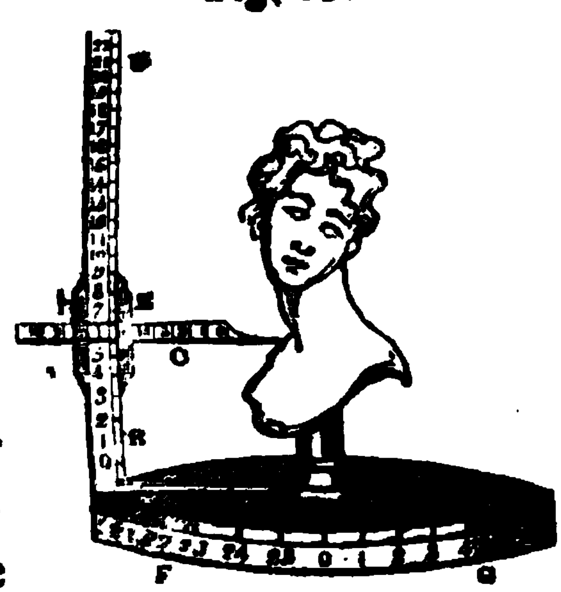
Sect. II. Principles conducting to a more refined analysis.

Art. 17. Example of a geometrical investigation.

The sculptor, for example, has to examine figures that are not readily analyzed by sections, but which admit of a convenient investigation by other applications of the principle we have described.

Let it be required, for example, to analyze a bust, fig. 59, with the view of preserving such a record of its parts as will enable the artist to form, at any time, an exact copy of the original.

Fig. 59.



Prepare, for this purpose, a circular stand, or table, FG, divided at its edge into any convenient number of equal parts: let a broad index, or radius AB, bearing a vertical graduated rod BD, play freely round a shaft fixed in the middle of the table. Attach to the vertical rod a slide E, that can be carried to any division of the rod, and secured there by means of a screw. Finally, let a graduated fillet C pass through the slide, and be capable of a motion towards, or from, the centre of the table.

With such an apparatus, the form of the bust can be readily examined.

For if we suppose the latter placed, immovably, on the central shaft, the three motions obtained, by turning the radius AB, elevating the slide E, and moving to or from the centre the fillet C, will enable us to bring the extremity of the latter in contact with any given point of the figure. Now, if we observe the division on the edge of the table to which the radius B has been carried; the division on the rod BD that corresponds to a known mark on the slide; and finally, the division on the fillet C, that corresponds with the edge of the vertical rod; it

is evident, that whenever the radius, the slide and the fillet are again adjusted to these divisions, the extremity of the fillet must "again" be in contact with the given point on the bust.

Whence, removing the latter, and substituting in its place a block of marble, we can ascertain, by adjusting the parts of the apparatus to these divisions, the extent of marble that must be cut away in order to arrive at the point within the solid that corresponds to the point examined on the bust.

Repeating these operations on all the prominent parts of the latter, we transfer those points to the marble, and obtain data that enable the artist, by the assistance of a well practised eye, to copy the original with great exactness.

In this process the "open figure" is formed by the three material rods AB, BE, and the portion of the fillet between the rod BE and the object: its inferior extremity is at A, the centre of the table, and its superior extremity corresponds to the tapered end of the rod, or to the point examined on the object. The results to be recorded for each open figure are, the divisions on the table, the vertical rod, and the fillet.

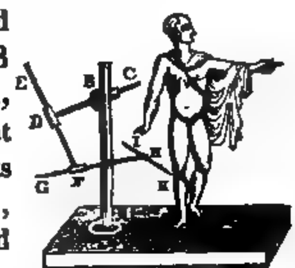
The dimensions and nature of the object are supposed, in this arrangement, to permit a certain degree of mobility; but, even with this condition, the apparatus is not applicable to every variety of form. Certain parts of a statue, for example, may be so placed as not to admit the approach of the fillet that is to determine their position; the arms, or other members, or the drapery, may be interposed, or lastly, although the contact takes place, it may be too oblique to admit of accuracy.

When such is the case, an arrangement may be em-

ployed that will afford us a third example of the use of "open figures" in analyzing the varieties of form.

The object in the arrangement alluded to, is placed upon a stand in which an upright pillar AB turns freely. A cylindric rod, CD, passes through an opening at B in this pillar, and carries at its extremity a second rod EF, which, in like manner, carries a third GH, and a fourth IK. The last being tapered to a point at K.

Fig. 60.



The rods pass through cylinders at B, D, F and H, in which, by a due application of force, they may be either slid or turned round their axes.

Now suppose the rod AB to be so turned, it carries with it all the rest of the apparatus, and along with other parts, causes a motion in the point K: if this motion is not sufficient to bring K into contact with the point to be examined in the figure, we may effect that purpose either by turning the several rods round D and F, or by sliding them through their several tubes, or by uniting, at the same time, all of these motions.

The scale on which the diagram is drawn does not permit the whole of the apparatus to be distinctly shown, but the motions we have described will be sufficiently understood from figure 61, where we have exhibited, on a large scale, the extremity of one of the rods, and the machinery by which

Fig. 61.

it permits, in the rod attached to it, the double motion we have mentioned.

Let us choose, for example, the rods or rather tubes CD and EF of figure 60.

The extremity D of the first of these tubes, denoted in figure 61 by A, passes *through*, and is secured to the inner of the two circular plates EF.

The outer of these plates is attached to a cylindric axis H, passing into, and turning in, the tube A.

A short tube D, secured to the outside of the same plate, carries the rod or tube denoted in figure 60 by the letters EF, and in figure 61 by CB.

As the inner surface of D, and the outer of CB, are not cylindric, the latter tube cannot be turned round in the former, but can be freely slid through it in the direction of the length.

The rod CB, and the edges of the plates EF are graduated.

This construction fully comprehended, the two motions to which we have alluded will be readily understood.

The first consists simply in sliding CB through the tube D; measuring the extent of the motion by the parts into which CB is graduated.

The second motion is produced by revolving, about their common axis H, the outer plate EF and the tube D attached to it. The extent of the revolution is here measured by the divisions on the edges of the plates; and when the motion has been completed, the possibility of any subsequent change is guarded against by turning a screw, G, that presses on the axis.

An apparatus of this kind placed at the extremities D

and F (fig. 60), will render EF and GH capable of being moved through any number of inches in the directions of their lengths, or of being turned through any number of divisions about CD and EF as axes.

The rods CD and IK have the sliding motion only, whilst the pillar AB has merely the motion of revolution. The two former slide through tubes at B and H, and the latter revolves in the socket at A, where the extent of the motion is measured by an index.

As these motions, when used at the same time, enable us to bring *the feeler*, or, extremity K, into contact with any given point on the statue, we have merely, in order to record the position of this point, to preserve the number of inches through which the rods were slid, and the number of divisions passed over in their revolutions.

When such memoranda have been preserved for all the principal points in the statue, the remainder of the operation, by which we transfer those points to an unhewn block of marble, will be identical with the corresponding process in the preceding case: the two methods of investigation varying merely in the different modes of arranging and recording the positions of the parts of an open figure; in the former process, this figure consisted of the sides AB, BC, and the rod C (fig. 59); in that which we have here described, of the sides AB, BD, DF, FH, and HK.

From these useful, but, comparatively, humble, investigations, we now pass to that great problem with a view to which our inquiries have been conducted. The dimensions and form of the earth were the questions proposed at the commencement of the present article as an illustration of the powers of geometry; and the proposi-

tions that have since occupied our attention were merely preparatory to this more extensive application of the science of form and magnitude.

The immense scale on which our operations have now to be conducted renders a material base altogether inapplicable; but the experience gained in the preceding inquiry has increased our powers of investigation, and will teach us how to proceed whilst endeavouring to overcome the difficulty alluded to.

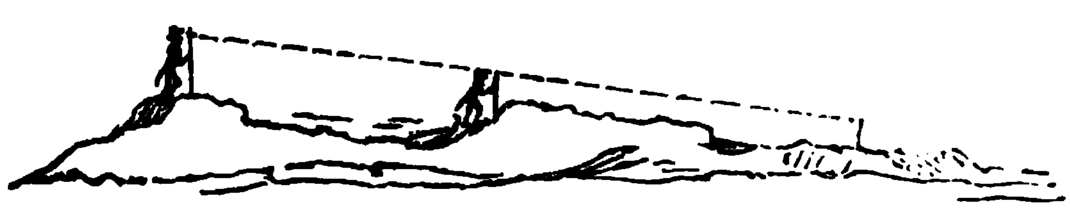
Whilst engaged in examining objects of small size, we made use of material planes to determine the figure. As the objects became larger we substituted for these material planes a number of material lines, arranged in one plane.

Pursuing this principle, we must now, where the object to be examined is of enormous dimensions, divest ourselves still further of material apparatus; and retain merely a few fixed points in the plane we wish to construct.

Nor will this be difficult: the optical principle that enabled us to dispense with one of the bases will enable us to dispense with both.

For suppose any two points very distant from each other, fig. 62, the summits of two small pyramids, for

Fig. 62.



example: let the eye be placed at one of these material points, and directed towards the other.

The line of vision is in this case the straight line between the material points.

Now imagine a third pyramid to be erected between the two former, and let its height be such as to bring the summit exactly into the line of vision. The summits of these three pyramids will then be in one and the same straight line.

In this manner innumerable points in the line, intermediate between the first and last pyramid, may be obtained: but the process can be carried further, and will enable us to obtain points beyond the last pyramid, notwithstanding the distance of this object from the eye may be the greatest that is compatible with distinct vision.

To determine the points here spoken of, it is merely necessary that the eye of the spectator should be placed at the summit of any one of the intermediate pyramids, whilst his attention is directed towards that which is more remote; the line of vision passing through the summits of these two pyramids, will coincide with the line which passes through the summits of them all, since only one straight line can be drawn through two given points.

But as the spectator has now approached the more distant pyramid, he can see distinctly objects that lie beyond it; and can ascertain when the summit of a pyramid yet more remote attains the line of vision.

By such a process, therefore, any number of points in a straight line may be obtained; and the line produced to any extent that is desired.

If this line is to serve as a base by which the figure and magnitude of the earth are to be measured; it must be continued to a considerable extent: two or three hundred miles is but a small quantity when compared with

a body of such great magnitude; it is however a more extensive base than we shall require; our present object is merely the illustration of a principle; and, hence, assuming a base of fifty miles, we shall seek in what manner to construct a plane that shall pass through it.

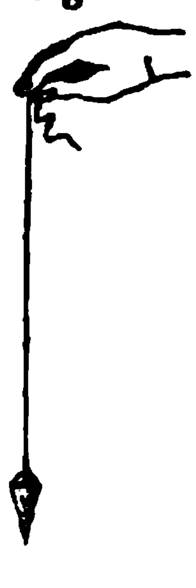
The material rods, or fillets, used in the preceding investigation will not assist us in that which now occupies our attention.

Not only would their great length become embarrassing, but, even if many intermediate stations were employed, it would still be difficult at such distances to look the edges of the fillets into one plane.

This difficulty may however be surmounted by having recourse, as on a former occasion, to a science foreign to geometry: on the occasion alluded to we extended our powers of geometrical inquiry by the assistance of a principle borrowed from optics, and in the present instance we may obtain a similar extension by appealing to a well known principle belonging to another department of physics.

Physics is a science of observation, and its principles are facts gathered from experience.

Fig. 63.



Now it is a fact gathered from experience, that "plumb lines" which are freely suspended will take directions perpendicular to the earth's surface; and it is also a fact of the same kind, that, when a number of such plumb lines are suspended from the same straight line, they will arrange themselves into one plane.

The summits of the pyramids are points in the same straight line.

If, therefore, we either suspended plummets from these points, or, in any other way, drew from the latter

perpendiculars to the earth's surface, the lines so drawn would lie in the same plane.

They would act the part, in short, of the straight fillets used in the measurement of a work of art. And if produced downward until they met the earth, their inferior extremities would be points in the intersection of the surface of that body with the plane we have constructed.

The method of proceeding will perhaps, after what has been said, appear sufficiently obvious; but as the measurements are to be made from the base line to the surface of the earth, it is necessary to perform the investigation in some place where that surface is regular and undisturbed. The violent convulsions to which the earth has been exposed have so deranged and broken up its external crust, that in many countries of considerable extent the true surface is no where seen.

In others, however, this surface is exposed to the eye in one vast, unbroken plain, that viewed from a distance appears merely a continuation of the ocean that washes its shores.

The bosom of the ocean, when unruffled by the wind, ought, indeed, on many accounts, to be regarded as the natural boundary of that solid body over so great a portion of which it is spread. And the base line of which we are speaking should therefore be carried along the sea shore, that we may measure at each station the distance from the base to the surface of the sea.

But in producing, according to the method above described, a right line of fifty miles in length, the pyramids, built up to the line of vision, will at remote distances become of great height, and thus introduce practical difficulties that would render the inquiry impossi-

Reviewing this analysis of the process we have pursued in investigating the form and dimensions of the work of art which has occupied our attention, we find the record of that investigation to consist—"first," of a model of the angle at which the fillet crossed the base; "secondly," of a graduated rule; and, "thirdly," of a table, expressing how many divisions of the rule correspond to the length of each fillet, and to each interval by which the fillets were separated.

But this simple method is equally applicable whatever are the dimensions of the object examined, and may be conveniently employed to record the process by which we proposed, page 43, to investigate the figure of the earth.

The distance between each station, the height of each station from the surface of the sea, and lastly the inclination, at each station, of the plumb line and base, are the facts to be recorded in the table. The remaining documents are the graduated rule and the protractor, with which the distances and the angles were respectively measured.

With respect to the last we may observe, that, as a standard linear measure is kept among the public records of most civilized nations, it will be unnecessary, unless extraordinary accuracy is required, to preserve the graduated rule used in the operation described. Every measuring rod, and, even, every carpenter's rule, is merely a copy, more or less exact, of the national standard: and if such were used in the operations described, we are at liberty, in any subsequent part of the process, to substitute for them other copies of the same standard.

The preservation of the protractor may also be dispensed with: for as that instrument consists, merely, of

the right angle—or one-fourth the space about a point, divided into a certain number of equal parts; an exact copy of it may be obtained at any time, provided, however, the number of divisions is recorded.

The table is, therefore, the only document required, and, to illustrate it yet further, we will add an example of such a table; supplying its contents from actual measurements that have been performed upon the earth, although according to principles somewhat different from those which the table is intended to explain.

No. of stations.	Distance of each station from the first; measured at the level of the sea.	Height of each station above the surface of the sea.		Angle between the plumb line and base.	
	Miles.	Feet.	Inches.	Degrees.	Minutes.
1	0	10	0	90	0
2	10	76	4	89	51
3	30	597	0	89	34
4	50	1658	4	89	17

NOTE.—The degree is the ninetieth part of a right angle, and the minute is the sixtieth part of the degree.

The remainder of the investigation, if pursued according to the method we have already employed, would require that we should delineate, from the data in the table, a picture of the section measured.

But as the immense scale on which the operations have been conducted renders it impossible to delineate a picture of the same extent, we must have recourse to the principle of "similar figures," art. 14, and construct on a greatly diminished scale a representation of the section in question.

This reduction is easily accomplished.

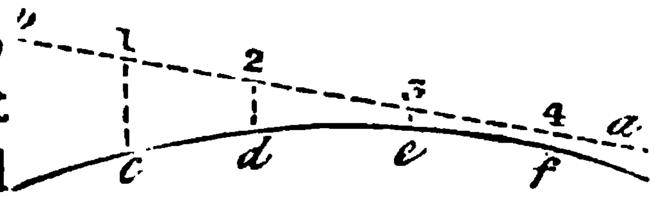
It is only necessary to substitute for the graduated rod by which the distances in the table were measured—a rule divided into the same number of equal parts, but,

each division on which is a very small portion of a division on the rod.

Let us suppose, for example, the divisions on these two graduated lines to have the proportion that a foot bears to a mile. The representation of the base of fifty miles would thus be reduced in the same proportion; and if a plate of copper, or other soft metal, were firmly set into a solid bed, and wrought into a smooth plane, eight or ten inches broad, and fifty feet long, the picture might be delineated upon this plane with considerable accuracy.

The process of delineation is, in every respect, the same with that described in arts. 11, 12. A picture *ab*, fig. 65, of the base line is first drawn, and a point, 1, assumed as the place of the first station. A distance from 1 to 2, of 10 feet, is then measured along this line, and, as a foot corresponds to a mile, the point, 2, so determined, will represent the position of the second station. The places, 3 and 4, of the third and fourth stations, are obtained in a similar manner.

Fig. 65.



This accomplished, a protractor is successively placed at each of the stations, and straight lines drawn that make with the base the angles mentioned in the table.

The lines represent the directions assumed by the plumb; and, consequently, if, from their intersection with the base, distances are measured, equal, respectively, to the heights of the stations above the surface of the sea, the points *c*, *d*, *e*, *f*, so determined, will be points in the intersection sought.

In the diagram we have referred to, the number of these points is very limited, but the process used to

obtain this limited number will provide as many as are required; and when a sufficient number has been obtained, the methods, already described, for passing a curve line through given points will enable us to delineate the intersection in question.

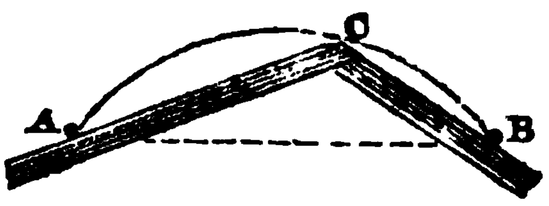
One of the methods alluded to consisted in a comparison of the curve sought with the material models of a series of known curves, art. 11. If such a comparison were made in the present instance, the arc of a circle would be seen to pass through all the points. The radius, however, of this arc would be so very great that to describe it with compasses would be impossible.

But other instruments are used as compendious series of circular arcs.

The trammels, 66, which are merely the two sides of a constant angle, compose an instrument of this kind.

They are made to slide in contact with two fixed pins at A and B, and the point of intersection, C, will then describe a portion of a circle.

Fig. 66.



Arcs of every radius may be formed in this way.

And if a series of such arcs was compared with the points delineated on the surface above described, they would be found to coincide with an arc whose radius was nearly four thousand feet.

Similar measurements performed in any part of the earth would give very nearly the same result.

And since a sphere is the only solid the intersection of which is a circle in every position of the dividing plane, we conclude the figure of the earth to be nearly that of a sphere, whose radius is four thousand miles.

Chap. I. Classification of the varieties of form.
Art. 18. Reflections on the preceding example.

18. The process of investigation that has led to this result will be found to consist of many portions, essentially different in their nature, and capable of being made the objects of distinct examination.

In some parts of the inquiry, the reasoning we have used is strictly geometrical, or conducted, art. 15, by "mental superpositions," and consequences that "necessarily" flow from them: whilst in others, such mental processes are superseded by facts drawn from induction, and by actual superpositions, tested by the evidence of the senses.

All investigations relating to the forms of bodies actually existing must be of the same mixed character: but a comprehensive system of classification will separate in such inquiries the portions that depend on distinct functions of the mind, and will treat of them as sciences apart.

It is the glory of algebra, the most perfect branch of logic, to distinguish in its investigations the train of reasoning common to a great many propositions, from that which is peculiar to each.

Such mental arrangements in the sciences correspond with the division of labour in the arts; and are productive of an incalculable increase of power.

Now if we look attentively at the preceding investigation, we shall discover in it these three distinct parts.

"First"—A purely geometrical analysis which has for its object to discover the relations of figures considered as existing only in the mind.

"Secondly"—A part whose object is the actual construction of lines—planes—and other figures.

"Thirdly"—A part which, connecting these two,

endeavours, by their assistance, to solve particular problems.

19. These parts will be found in every such inquiry, and give rise to three sciences.

The science which treats of the first is termed "Theoretical Geometry"—or, simply "Geometry:" it is to this branch that we have in the present treatise more particularly assigned the term "Inductive Geometry:" and the reader will recollect that it has been defined in art. 15, as—"the science which teaches us to classify the varieties of form, and to discover their hidden relations by a mental comparison of the figures so arranged."

The sciences treating of the second and third branches are usually confounded together under the term "Practical Geometry." They are, however, very distinct. The former treats of the material instruments used in practical geometry, and the latter of the use of them. The former—that is the second branch in this triple division, ought to be distinguished by a name signifying the construction of models; for every instrument which the mathematical instrument maker constructs is merely a model, or a series of models, of forms already existing, art. 6.

This branch we shall consider in the present treatise under the head—"Of the models used in the application of geometry."

The third, or last, of the branches above enumerated, arranges itself into as many distinct subdivisions as there are arts and sciences to which geometry can be applied.

But on attentively considering these subdivisions, we observe a peculiar characteristic to distinguish the geo-

metry applied to the arts, from the geometry applied to the sciences.

In the arts those methods are to be regarded as the best which accomplish with sufficient accuracy, and the greatest economy of time, the particular object in view.

But in treating of the sciences other considerations govern the choice of means. The economy of time is studied, not with relation to its subdivisions, but with regard to periods of great extent: increased accuracy becomes a source of new discoveries: and as every part of a science is viewed in relation to the whole, the principles borrowed from other sciences must be equally comprehensive.

From these causes the geometry that is most useful in the arts will not be so refined and abstract as that employed in astronomy or physics. It partakes of the subject to which it is auxiliary; and the geometrician becomes an artist surrounded by the materials of his profession.

This last subdivision of the branch we are treating has been termed by Monge "Descriptive Geometry," and is thus defined by him—"Descriptive geometry has two objects: first—to give methods for representing on a surface that has but two dimensions, namely, length and breadth, all the bodies in nature which have three—length, breadth, and thickness; provided, however, these bodies are such as can be rigorously defined.

"The second object is to give the method of recognising the forms of bodies from exact descriptions of them; and to deduce all the truths that result both from their form and their respective positions."

A slight sketch of the leading principles of descriptive

geometry will be found at the end of the present volume, and in another part of the work we shall treat of the applications of geometry to surveying, to geography, and to astronomy.

The full details of these subjects would be improper in a work that is professedly an introductory treatise of pure mathematics: but the same objection does not apply to the theoretic branch, to which we shall now return.

20. Theoretical geometry, we have seen, is merely a very refined method of classing the varieties of form, and so arranging them, that, truths proved of an individual figure shall be equally applicable to innumerable other figures of similar forms.

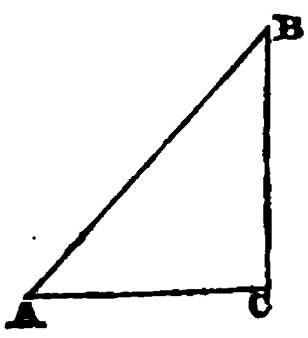
Its demonstrations are independent of the physical objects that present to our senses the forms we are investigating.

Our models are neither of wood, nor metal; they are disembarrassed of every quality but figure.

Existing only in the mind, they can be enlarged or diminished, varied, or compared, with the same facility that other ideas are made to pass through the mind.

Truths obtained by actual measurement are properties of that figure only on which the measurement is made, but the truths of geometry belong to an infinite series of figures.

Fig. 67.



If we are told, for example, that a person constructed a right angled triangle and made three "geometrical" squares; one square whose side was equal to AB, another with the side AC, and a third with the side BC: if we are told, further, that he cut the smaller squares into a num-

ber of parts, and fitting them together, as we fit the detached parts of a child's puzzle, discovered the parts of the smaller square to cover, when taken together, the area of the larger: the fact so discovered would be curious and interesting, but the question immediately suggests itself—"is this property true of right angled triangles of every size and shape?"

Right angled triangles are not all alike!—on the contrary they admit of infinite variety—and what is true of any particular triangle of this species, may be false for another right angled triangle of a different form. A method of proceeding, then, which examines individual figures, and demonstrates its truths of them only, is, at best, unfitted for the purpose of those arts and sciences whose truths are of a general nature.

But not only would all practical methods of measurement fail on the grounds here mentioned—that is—because we could measure, by their assistance, only a few out of an infinite number of figures—they would prove insufficient even with respect to the figures that we *did* measure: the truths of geometry are applied to the phenomena of the heavens; what we measure here on diagrams of a few feet or inches, is there tested on a scale that knows no bounds. We thus want, not an approximation to truth, but truth itself; not a measure relative to our imperfect senses, but a necessary consequence of the immutable relations of space.

How absolutely necessary such results are for the applications of geometry to those great problems that afford the means of intercourse between nations, or unfold the nature of the universe we inhabit, has been partially illustrated in the only geometrical investigation that has hitherto engaged our attention.

Sect. II. Principles conducting to a more refined analysis.

Art. 20. Advantages peculiar to abstract geometry.

With rude and very imperfect means we endeavoured to discover the form and magnitude of the earth upon the principles adopted in measuring a work of art.

The investigation, adapted only to the first infancy of the science, led us gradually to refine the yet more imperfect means that we had previously discovered; and by enlarging our notions respecting the construction of lines and planes, and the methods of recording investigations, enabled us to escape, in a great measure, from the work shop of the artizan.

But the whole investigation was based upon a principle that abstract geometry can, alone, establish.

The principle, namely, of symmetry: of the independence, art. 14, of form and magnitude.

According to this principle, the ratio obtained by comparing any two parts of a figure, will be the same as the ratio obtained by comparing two corresponding parts in any other figure of the "same form." The circumference of a circle, for example, will contain the radius as often in a circle of large dimensions as in one of small.

This principle is, indeed, so very obvious, that it might perhaps be assumed as an axiom, that is, a truth that does not require the assistance of demonstration. But the diversity of minds is so great that truths which are axiomatic to one person, may appear very obscure and even doubtful to another. The previous habits of mind, and the class of objects with which the senses have been familiar, produce astonishing diversities in this respect: an artist, whose daily pursuits render particular portions of some complicated machine the objects most familiar to his thoughts, will acquire an intuitive perception of the principles that govern their action. But the acqui-

sition of this tact is not accompanied with a power of demonstrating the truths so perceived; the contrary is usually the case; and an ingenious and skilful workman is often incapable of explaining the laws that regulate the machinery invented by himself.

21. But if, in the arts, such are the necessary consequences of an incomplete perception of first principles, the want of this elementary knowledge is far more extensively felt in the sciences: the vast field which the latter embrace renders it necessary to proceed with the utmost caution—a single error may vitiate a whole science—and hence it is not sufficient that fundamental propositions should be perceived at the moment of considering them; they must be placed beyond the reach of subsequent doubt. The science that is to assist and direct so many others must itself be based on unquestionable foundations; and cannot be regarded as complete unless every proposition is traced back to a few elementary truths, that appear incapable of further analysis.

The independence of form and magnitude on which we founded the investigation of art. 17, is a principle that requires this ultimate analysis; and hence, were the investigation alluded to in other respects unobjectionable, it would be incomplete in this.

That investigation, however, the reader is aware, was not chosen as an example of the method that geometricians would adopt in examining the form and magnitude of the earth: it was intended to exhibit the power of the elementary principles of classification previously developed.

In accomplishing the illustration of these principles, we had occasion to notice the additional power and ac-

curacy that would be obtained by combining them with others.

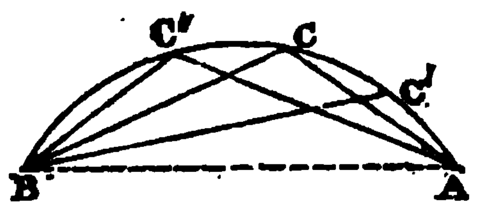
The idea indeed of representing all figures by means of their most conspicuous points had occurred in art. 4, and in the application of this principle consisted the chief advances that were made in prosecuting the investigation of art. 17; but although the principle had occurred, the means of carrying it into practice had not occupied much of our attention; and was only partially developed in the inquiry alluded to.

A moment's consideration will convince us of this fact, and demonstrate at the same time that it is to the discoveries of abstract geometry that we must look for the improvement of that which is practical.

The instrument, for example, used at the conclusion of art. 17, to construct the series of large circular arcs there mentioned, depends upon a property of circles that can only be fully established by the mental demonstration of abstract geometry.

The property in question asserts that "all angles inscribed in the same segment of a circle are equal."

Fig. 68.



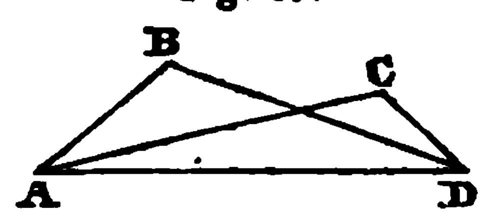
By a segment of a circle is meant a portion cut off by a straight line, or "chord," AB.

And an angle is said to be inscribed in a segment, when the sides pass through the extremities of the chord, and the vertex is in the circumference of the circle.

22. But if this property can, by any abstract process of reasoning, be generally established of all circles, it will follow that we may dispense altogether with the cumbrous series of material arcs which the property

alluded to was employed to construct. We may ascertain the nature of the curve investigated by the points in it which the investigation has made known, and without the necessity, either of drawing a curve through those points, or of comparing them with the materials models of such a line.

Fig. 69.



All that is necessary is to construct on the plane surface wherein the points lie, angles that have their sides passing through the two extreme points, and their vertices in the points intermediate.

If the angles formed in this way are equal, the points lie in a circle, otherwise not.

This process, it is true, fails to inform us of the radius of the circle that passes through the given points; but it appears not improbable that some other property of the lines drawn in a circle would also accomplish that object; and, accordingly, in another part of the work we shall see that such is the case.

This simple idea, therefore, of determining the properties of figures from those of points that lie in them, has, after many successive steps, disencumbered us of the greater part of the physical machinery which embarrassed the great problem considered in art. 17.

The application of two other principles, already developed, will complete this series of improvements, and render the investigation incalculably more refined than the method which first suggested itself.

The two principles alluded to are derived from the symmetry of figures—from that independence of form

on magnitude which has already enabled us to accomplish so much.

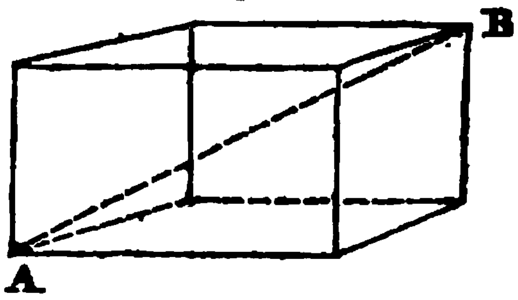
From this truth, when once completely established, it will follow that—with respect to a rectilinear figure—a figure, namely, whose "edges," art. 3, are straight lines—the form will be determined when the angles are known.

How completely the form depends on the positions and lengths of the edges has been seen in art. 4. But the principle of symmetry goes further, and asserts, as will be hereafter demonstrated, the form to depend altogether on the angles.

The angles here spoken of are not, however, merely the inclinations of the edges; but the inclinations of all right lines that can be drawn from one corner of the figure to the other corners.

These lines include the edges together with lines which, if the figure is solid, fall within it. The line AB, for example, drawn between the solid angles A and B of a parallelopipedon, falls wholly within the solid.

Fig. 70.



23. If then in investigating many figures we were to measure all the angles here mentioned, the form would be completely determined; and the principle of superposition would then show the magnitude to depend on the length of any one of the sides, or edges. The process used to demonstrate this fact is in every respect the same as that of art. 15: it proceeds by assuming two figures having, in each, the corresponding angles, and one of the sides, the same: and from these data we prove

that either of the figures can be made to coincide, line for line, with the other:—that in imagination such a figure can be so placed as to fill the same space:—and, in short, that it is in our power to render the two figures identical.

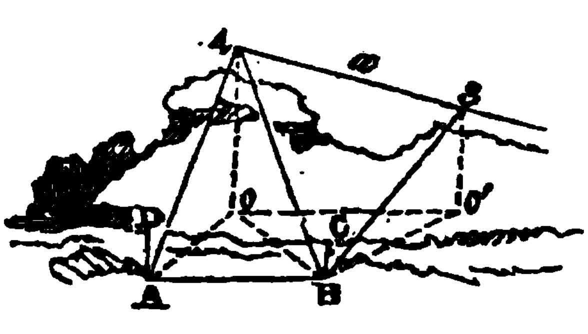
The demonstration of this property cannot be accomplished until we have made considerable progress in the analysis of form; its application to the problem of art. 17 may be readily exhibited.

24. The demonstration appears to rest on two results, that flow immediately from the property in question. It is a corollary from this property, that a knowledge of the length of a single line in a figure determines the magnitude of the whole.

It is also a corollary that, if, in conducting the investigation of art. 17, the base line and the several altitudes are made parts of a single figure, their lengths may be estimated from the length of some one line in that figure.

The whole linear measurements involved may therefore be reduced to the measurement of a single line: and, as the position of this is arbitrary, it may be chosen where the nature of the ground or other accidental circumstances render most convenient.

Fig. 71.



Let it be assumed, for example, that 4 and 3 were two of the stations, and a a portion of the base line.

Let also AB be a line measured along the shore, and AD, BC perpendiculars erected there.

The altitudes of the stations above the level of the sea

will be measured by vertical lines 4 O, 3 O', that fall within the eminences on which they stand.

Unite, in imagination, the points O and O' with A and B.

The lines A 4, B 4, B 3, A O, B O', $O\,O'$, O 4, O'3 and a will form the edges of an imaginary solid; to determine the form and magnitude of which, it will only be necessary, according to what has been said, to measure the side AB, and the angles formed by the edges of the solid, and by all the other lines that can be drawn from point to point of it.

Instruments have been contrived for measuring these angles with great accuracy.

The angles A 4 O, B 3 O', cannot be measured by directing the line of vision towards O and O'; for those points, lying in the interior of the eminences on which the stations are erected, cannot be seen: but here also, abstract geometry comes to our assistance, and by demonstrating the angles in question to be respectively equal to 4 AD and 3 BC, removes the difficulty.

By continuing the process the other stations may be made points in the figure, and their altitudes brought to depend on the length of AB.

The length of this last, it appears then, determines that of every line in the figure; and as it requires to be only a few miles in extent, and may be chosen on a level spot, such a method of computation will be capable of incomparably greater accuracy than could be attained by the method adopted in art. 17.

Nevertheless, a difficulty remains unsolved.

It is the construction of the reduced model, or the model whence the figure investigated is to be determined.

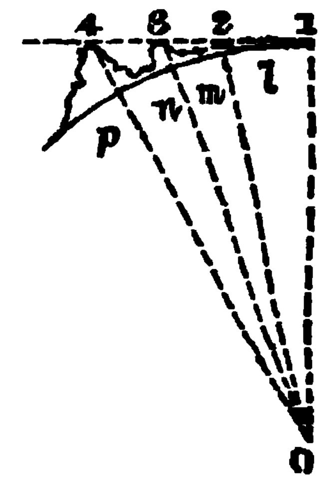
Chap. I. Classification of the varieties of form.

Art 25. The investigation of art. 17 conducted by assuming, as known, the figure which is the object of inquiry.

25. But the principle that has so advantageously replaced the actual measurements of the base line,* and of the altitudes of the several stations, may be used to advantage in the remainder of the operation; and will not only surmount the difficulty to which we have alluded, but enable us to dispense with the construction of the model in question.

To attain this purpose, it is only necessary to "assume" the form of the earth to be that of a sphere, and to regard the centre of the latter as a point in the solid figure whose relations we are investigating.

Fig. 72.



Imaginary lines drawn from each station to the centre of the earth will effect this connection: and as it will be shown in our future researches that the angles at O are immediately dependent on those at 1, at 2, at 3, &c. which last have been already measured; it follows that we are acquainted with all the angles of the figure formed by the straight lines whose pictures are delineated in the diagram.

The magnitude of this figure will therefore depend on that of any one of its parts.

But the lengths of the base 1, 4, and of each part of the base, have been calculated, and made to depend on the measured line AB, fig. 71.

Hence it follows, that each of the lines O 1, O 2, O 3,

* The line from which the curvature of the earth is estimated is that, it will be recollected, which is here termed the base. In surveying, the term is more usually applied to the line actually measured, or which corresponds to AB.

&c. must also depend on AB, and admit of an exact appreciation by the rules which contain the mutual dependence existing between the lines of rectilinear figures.

These rules, we again repeat, are to be the objects of our further research, and will be established when we are prepared to treat in a more methodical manner the principles of abstract geometry; but in point of fact these rules are a subordinate part of the question we are considering; and supposing them established, a question immediately suggests itself, how, from a knowledge of the lines just mentioned, are we to determine the form of the section l m n p?

A very obvious reflection will enable us to answer this question. It will be perceived, that since we assumed the section to be a circle, the length of each of the lines, ol, om, on, &c. must be the same, these lines being radii of that section.

Taking, therefore, the known altitudes il, 2 m, 3 n, &c. from the calculated lengths of O i, o 2, o 3, &c., we must observe whether the remainders are equal, and should this appear to be the case, the truth of the hypothesis, or, in other words, the circular form of the section, is established.

Much, indeed, would yet be wanting to prove the several steps of the process with geometrical accuracy, but enough has been accomplished to exhibit the powers of geometry, and to demonstrate the immediate dependence of the practical branches upon those which are abstract.

26. A single observation, however, remains, before we conclude the illustration that has been afforded by the

preceding problem, of the successive stages through which geometry may be supposed to have passed.

Of the two principles mentioned in art. 25, the second has yet to be explained.

Both, we have remarked, are founded on the independence of form and magnitude; and both ought to be regarded as only particular methods of expressing that truth.

There is, however, another view that can be taken of the second principle; and as this method of considering the subject possesses many advantages, we shall for the future adopt it.

The forms of rectilinear figures depend, we have seen, upon the lengths of their sides, and the angles at which the latter are mutually inclined.

Let us recall the principles upon which these two species of quantity are measured.

The lengths of lines must have reference to some other line that is taken as a standard; and the quantities of angles are measured by their relation to the whole plane space about a point.

The ratios, then, of either of these quantities to the standards that measure them, will be just expressions of the quantities themselves.

But as the ratios in question are numbers, it appears that every relation of form and magnitude—or, according to what we have hitherto seen—every relation of the form and magnitude of rectilinear figures—will be reduced to a relation of numbers.

The rules which guided our researches concerning the properties of number, will thus become subservient to the present inquiry; and every improvement that re-

warded the diligence bestowed upon the former, will be a step in the acquisition of the latter science.

27. With these remarks we shall close the preliminary investigation that has occupied our attention in this division of the work: but, prior to entering on a more methodical inquiry, it will not be amiss to state briefly the elements to which we have ultimately traced the relations of form and magnitude.

The elements peculiar to geometry may be considered as among the simplest ideas we possess. They are few in number; and neglecting certain notions about continuity, may perhaps be reduced to three—place, direction and distance.

The first steps beyond these simple ideas have an immediate reference to them, teaching to designate by signs the places spoken of, and to obtain proper measures of their directions and distances.

For science, which has grown out of art by refining and extending its processes, and rendering the operative part of them purely mental, has been unable to dispense with many conceptions having their origin in the ruder operations of the measurer of land, or in the labours of the artizan. The pegs and stakes used by the former to mark the most prominent places in his fields, probably gave birth to mathematical points, symbols used for the same purpose in the mental measurements of the geometrician.

The size of the points is neglected in either case; it is omitted by the surveyor because the nature of his work does not make account of such small quantities; by the mathematician, for an analogous reason—the sensible

points he employs are regarded as symbols of position, that *ought not* to be considered as existences or magnitudes.

In this way we pass by an easy transition, from an acquaintance with external objects, to the ideas of place, direction and quantity. We reason on the parts of space, which are every where alike, by making use of something adventitious to them—of objects, course and rude in the first infancy of science, and gradually diminishing in size as it becomes necessary to indicate with increasing accuracy the *place* of which they are the signs: this process may be continued without limit, until, neglecting the magnitudes of the symbols, and depriving them, as far as possible, of physical properties, we are led to the abstract notion of mathematical points, signs that are not regarded as having a real existence.

The comparison of two points, the only idea we have of direction, will again lead to a similar abstraction—*the straight line*, or the symbol of space that has every where the same direction.

And following this process, and passing from the direction of points to the distance that separates them, we obtain the notion of *quantity*, and its correlative—*measurement*.

The *distance* of the points is the straight line which joins them. And we estimate the quantity of the line by a continued superposition of its own parts, or the parts of some other line similar to it—a process also borrowed from the operations of art, and forming the primary notion involved in every species of measurement; operations which have their origin in the same humble source: whether the surveyor carelessly stretches his line over the asperities of the ground; or the philosophic

artist exhausts all the refinements of ingenuity in adjusting the microscopic divisions of his *standard unit;* or, lastly, the geometrician *conceives* the superposition of his imaginary lines, and *proves* the *possibility* of their coincidence, the leading idea is in all these cases the same: yet how wide is the step made by the latter in substituting the possibility of performing an operation for the operation itself; and thus replacing the limited testimony of the senses by evidence purely mental!

And, as, from the comparison of points situated in one direction we obtained the notion of distance, so, by the comparison of two directions we form the idea of an angle; an idea altogether distinct from the former, but, like it, capable of measurement by the superposition of its own parts.

Symmetry might be classed with the elements of position: but as it seems a combination of the ideas already enumerated with that of *order,* which is common to our conceptions of time, space and number; there would perhaps be as much impropriety in ranking it with the elements peculiar to geometry, as in referring *equality* to the same source.

A similar observation will apply to our ideas of *continuity* and *infinity,* both of which seem to involve the conceptions of motion and time; they must be classed among those obscure ideas that all minds perceive with difficulty, and yet that perhaps all perceive alike. The reader may be surprised to find in a science having such pretensions to accuracy, the foundation of much that is essential referred to ideas concerning which no distinct notion can be formed. But it may often happen that, whilst the primary idea is obscure, many of its relatives are distinct; and enter so extensively into our reasoning,

as to render it impossible to discard them. Where such is the case, we should lay the difficulty fairly before the student, and let him appeal to his own mind for the degree of evidence he finds there concerning it. A violation of this simple rule led the ancient geometers into erroneous views concerning the use of definitions and axioms; errors that are unavoidable where too great weight is placed upon the dignity of science.

The infinite of space seems always to require a reference to eternity, the infinite of time. If we conceive a point as receding uniformly, and for ever, from another, we have perhaps as distinct a notion of quantity *infinitely great* as it is possible for finite minds to acquire.

But we can also suppose two points uniformly to approach each other, and whatever distance is assigned to them at a given moment, we can suppose a time when, by their approach, this distance is diminished to one half; another time when the remaining distance is also halved; and continuing the bisections we continue to decrease, at a uniform rate, the distance of the points: nor is it possible, on one hand, to set a period to the operation, nor on the other to assign quantities so small that the distance between the points shall not at last become less than, these, the least quantities perceptible to the mind. Such quantities are said in mathematics to be *infinitely small*, and although regarded as small without limit, are considered as retaining all their properties but magnitude.

It will follow from what has preceded, that the terms nothing and infinitely small, though often confounded, are not identical; the latter always conveying in mathematics the idea of an actual existence, whose quantity is neglected; nor shall we frequently err by extending the

same signification to the *zero* of mathematicians, which rarely means non-entity.

The reader will now, it is hoped, more perfectly comprehend the ideas intended to be conveyed when points, regarded as symbols of place, are said to be without dimensions—straight lines, the boundaries of angles, as length without breadth—and surfaces, the boundaries of solids, as length and breadth without thickness. These ideas, he will perceive, are abstractions formed from existing objects, to facilitate the process of classifying their forms.

The principles on which this process ought to be methodically pursued must now be considered, and will occupy our attention in the second division of the work.

FIRST APPENDIX TO PART I.

OF SOME SIMPLE FORMS.

THE ancients, in the attempts which they made to arrange the varieties of form, distinguished rectilinear figures into the "regular" and "irregular," subdividing them again according to the number of their sides.

A regular figure is one that has all its sides and angles equal.

The regular plane figures may have any number of sides, but the regular solids are more restricted.

Among the simplest examples of the first class we distinguish,

Fig. 72.

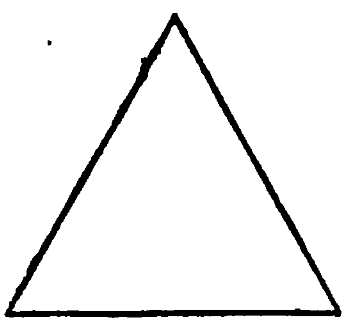


Fig. 73.

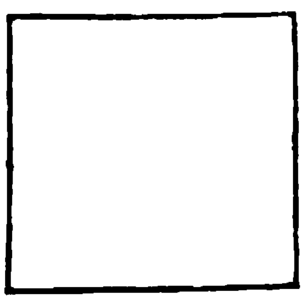


Fig. 74.

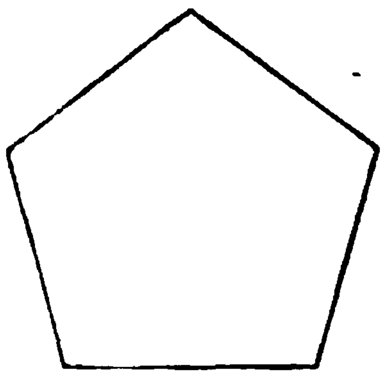
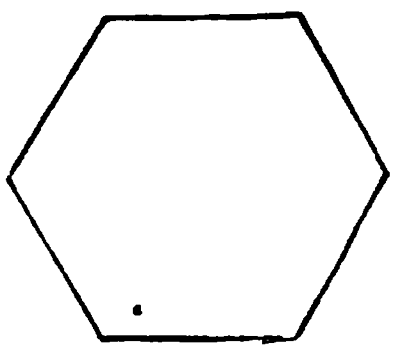


Fig. 75.



The equilateral triangle, which has three equal sides, fig. 72.

The square, which has four, fig. 73.

The pentagon with five, fig. 74.

And the hexagon with six equal sides, fig. 75.

First Appendix to Part I.

Of some simple forms.

The regular solids are only five in number, namely,

Fig. 76.

The tetraedron, or regular triangular pyramid, having four triangular faces, fig. 76.

The hexaedron, or cube, having six square faces, fig. 77.

Fig. 78. Fig. 79. Fig. 80.

The octaedron with eight triangular faces, fig. 78.

The dodecaedron, with twelve pentagonal faces, fig. 79.

And the icosaedron, with twenty triangular faces, fig. 80.

Models of these solids may be formed by previously developing them on a plane surface.

Fig. 81.

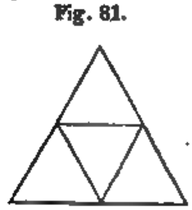
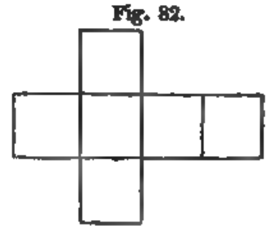


Fig. 82.



The development of the tetraedron is an equilateral triangle, fig. 81, and if this is divided into four equal

and equilateral triangles, each of the latter will represent the development of one of the faces of the tetraedron.

Hence, to form a model of that solid, it is only necessary to cut in pasteboard an equilateral triangle, and to divide it by sections that do not pass through the substance of the board, into the partial triangles represented in fig. 78. The outer triangles turned upon their bases, as upon hinges, will meet in a point and form the surface of the tetraedron.

Fig. 83.

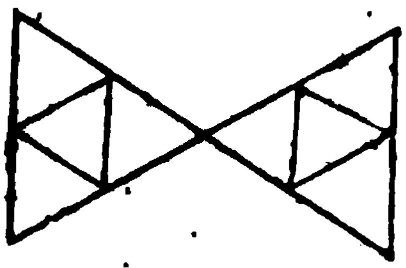


Fig. 84.

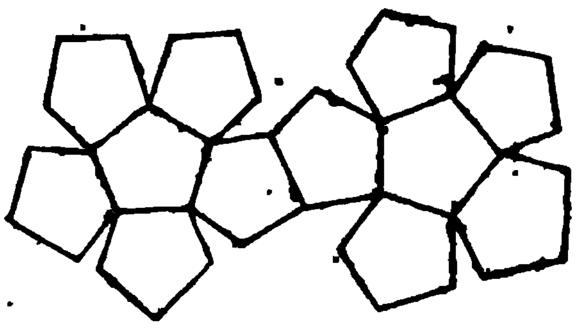
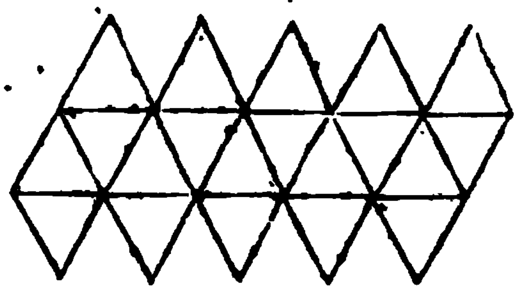


Fig. 85.



The figures 82, 83, 84, and 85, represent developments of the cube, the octaedron, the dodecaedron, and the icosaedron, that afford similar facilities for the construction of the models of those solids.

SECOND APPENDIX TO PART I.

PRINCIPLES DISCUSSED IN PART I. ARRANGED UNDER THE FORM OF QUESTIONS.

1. Whence is a knowledge of *form* derived?

2. The varieties of form are infinite in number, and appear to pass into each other by insensible gradations; yet, mathematicians, in establishing a geometric analysis, have proceeded on a principle that both enables them to refer to any variety of form, and renders a separate examination of each variety unnecessary. Explain the principle here alluded to.

3. On what principles are the varieties of form classed?

4. The comparison of figures may be reduced to relations yet more general, and which dispense with the necessity of material and solid models: what are these relations?

5. The comparison of solid figures may be reduced to the relations of pictures drawn on a plane surface: by what process is this accomplished?

6. How are geometric investigations performed?

7. When geometrical investigations are performed practically, it is necessary to be in possession of extensive series of certain elementary figures: explain the arrangement of some of the most simple and compendious of these series.

Second Appendix to Part I.

Principles discussed in Part I. arranged under the form of questions.

8. Known figures may be analysed into detached lines and angles, as simple elements; and may be reconstructed by placing these elements in their former position: but a more important problem arises, when, from certain of the parts, or simple elements of a figure, it is required to determine the remainder: how is this problem resolved?

9. Two species of arrangement are used in geometry—the first has for its object to classify the varieties of form—the second, to employ, in the analysis of more complex figures, the elementary forms already classed: explain the principles on which these two species of arrangement proceed.

10. Explain the method by which angles are compared.

11. Explain the compendious arrangement of angles that is known under the name of the "protractor."

12. What kind of figures are the ellipse, the hyperbola, and the parabola? Explain the compendious arrangement of these figures made use of by the ancients.

13. In what manner are arbitrary or accidental figures delineated?

14. What is that principle of arrangement which is common to all figures, and which distinguishes form from magnitude?

15. Prove that a relation among certain lines may necessarily involve a relation among others.

16. Explain the nature of a geometrical investigation, and show in what it differs from an investigation conducted practically.

17. Explain the different kinds of reasoning used in art. 17; distinguishing the processes that belong to pure geometry.

Second Appendix to Part I.

Principles discussed in Part I. arranged under the form of questions.

18. What are the advantages peculiar to abstract geometry?

19. What are the principles on which geometrical investigations should be conducted?

20. How are the relations of form and magnitude reduced to the relations of number?

21. What are the elements peculiar to geometry?

22. What is the idea intended to be conveyed when a line is said to be without breadth, and a point without magnitude?

23. What essential distinction is there between the measurements of abstract geometry and those of art?

PART II.

DETERMINATE ANALYSIS.

PRELIMINARY REFLECTIONS.

Geometry treats of the relations of place, points are the symbols of place, geometry treats then of the relations of points.

INQUIRY SUGGESTED BY THESE REFLECTIONS.

An examination of the most obvious relations of a finite number of points.

CHAPTER I.

FIRST PRINCIPLES OF THE SCIENCE OBTAINED FROM THE RELATIONS OF A FINITE NUMBER OF POINTS.

N

SECTION I.

OF QUANTITY.

Relations of two points—when direction is not regarded, the relations of quantity are the same with those of number.

28. The comparison of two points affords the ideas of a distance and a simple direction, and as things so dissimilar do not combine, these are the only relations which this comparison offers.

29. It has been remarked in the preceding pages that the calculation of quantity by the geometrician, and the process whereby the carpenter estimates the proportions of his work, have their foundation in the same elementary idea—an humble appeal to experiment is the source whence either derive this evidence; but it has also been remarked, that with this common origin, all analogy ceases between the two methods. The artist applies his standard measure as well as the nature of the object will admit, and trusts to observation for their coincidence. But the geometrician sets himself above the imperfection of his senses, and appeals to *mind* for the *process* as well as the result. Conceiving one magnitude applied to another, and abstracting, mentally, the obstacles which

prevent their coincidence, he observes the number of times that one contains the other, and analyses the quantity of this last into that of the first line and of *number*.

In like manner, if the first line be applied in succession to many others, the quantities of the latter will be made to depend upon that of the line applied, and the number of times it is contained in them. Thus, if this line is, respectively, contained in several others, 3, 5 and 9 times, we have an exact idea of the length of these lines, by calling to mind the length of the first, in conjunction with the numbers 3, 5 and 9. The process is not only analogous to that of division in arithmetic, but constitutes one of the sources whence our idea of division is obtained.

It follows, that if we agree to represent the line used as a measure by the letter A, the lines containing this measure 3, 5 and 9 times, will be represented by the expressions 3 A, 5 A, 9 A; where the symbols 3, 5 and 9 are signs of number, and A recalls to mind, simply, the idea of a particular extension. But whilst that extension continues the same, that is, whilst we employ the same common measure to compare the lengths of lines, there is some advantage in considering this measure as the *unit of length*, and in making its symbol the same with the unit of number; since, by so doing, the expressions A, 3 A, 5 A, 9 A become simply 1, 3, 5 and 9.

The method we have followed has enabled us to reduce the relations of extension to those of number, and even at this early period of our progress, we have obtained an advantage less perfectly possessed by the ancient geometry. The rules of algebra, from which this advantage is derived, teach us to denote numbers that are arbitrary,

or unknown, by letters of the alphabet, and in this way we can as readily express a line whose quantity is an object of research, as we could one whose length is expressly given.

Hence, when we meet with such an expression as the line a, we have merely to put it under the form $\frac{A}{1}=a$, in order to perceive that it signifies a line which contains the linear unit a times.

And in the same manner we can interpret such expressions as the line a^2 $a^{\frac{1}{2}}$, or a line denoted by the more complicated expression $\sqrt{a^2+x^2}$; for forming an equation, as before, we perceive these combinations of algebraic characters to denote the number of times which the linear unit is contained in the lines they represent.

Combining what is here said, with the remark in the first paragraph of this section, it appears that problems involving only the relations of two points, or of quantity without direction, can be reduced to numerical propositions, and be solved by the rules of arithmetic and algebra.

Let it be required, for example, to find a mean proportional between two straight lines: by combining these lines with the unit of linear measure, we are enabled to express them by known numbers, a and b; and assuming x to represent the number of times which the mean proportional sought contains the linear unit, we derive the equation,

$$x^2 = ab;$$

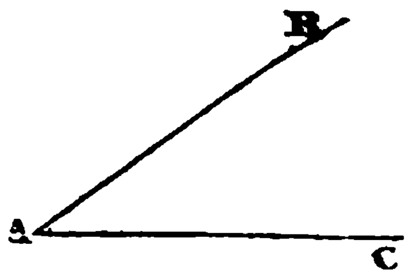
the solution of which determines the line sought.

SECTION II.

OF CLOSED FIGURES—CLOSED SOLIDS—AND THEIR RELATIONS.

Relations of three points: idea of an angle results from comparing the directions of three points—idea of a triangle obtained from the same source—relations of four points—straight lines that cross without meeting—idea of a plane obtained from the relations of four points—three points, or two straight lines, that are in a plane, suffice to determine its position—planes mutually inclined intersect in a straight line—measure of their inclination—relations of many points—notation to be used—closed figures—angles about a point—unit of angles—opposite, or vertical angles are equal—closed solids—solid angles—their unit—geometric analysis conducted by closed figures or solids.

Fig. 86.



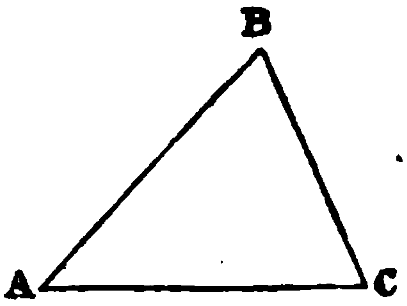
30. Comparing the directions AB of the points A and B, with the direction AC of the points C and A, we form the idea of an angle BAC.

And observing that every point in AB, or AB produced, has the same direction with re-

spect to A; and that every point in AC, or AC produced, has also the same direction with respect to A; we perceive the angle BAC to be independent of the distances AB and AC.

31. The angle BCA results from comparing the directions BC, CA; the angle ABC from comparing AB and BC.

Fig. 87.



And thus, from a comparison of all the points two and two, and all their directions two and two, we obtain three directions—three angles—and their arrangement.

This arrangement is termed a *triangle*.

32. Since the number of distances is obtained by combining the distances two and two, the relations of four points will involve six distances and fifteen angles.

The former may be estimated by considering three distances as connecting the point A with the other points, B, C and D; and three as connecting B with A, C and D; nor will it be necessary to carry the process further, since the distances connecting C and D with the remaining points, are the same with those reckoned from A and B, taken in a contrary direction.

Fig. 88.

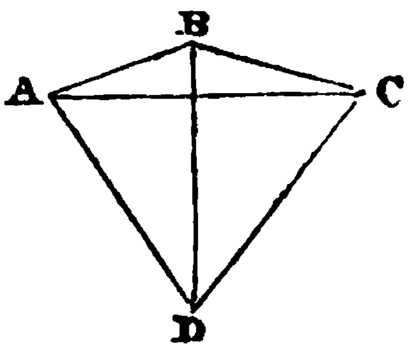
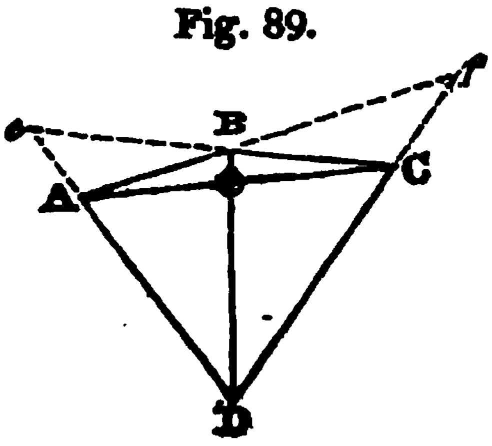


Fig. 89.

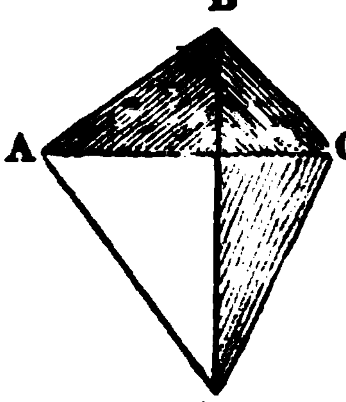


Of angles there are,—twelve about the four given points, three about each—and three that are not about the given points, namely, two at *e* and *f*, formed by producing

the lines DA, CB; and AB, DC;—and one at O, formed by the lines AC, BD.*

33. But reflecting on the position of these points, it will immediately be seen that A, B, C and D are placed at the angles of a pyramid, (fig. 90), and, consequently, that several of the lines we have been considering, cross without intersecting or meeting each other. This may be the case with AC and BD, regarded above as meeting in the point O; with BC and AD, regarded as meeting at *e*, and finally with AB and DC.

Fig. 90.



34. These lines, that admit of being produced indefinitely, without having a point in common, suggest an idea only inferior in simplicity to those of the straight line and the angle: for reflecting on the remark made in the preceding article, that BD and AC cross each other without meeting, we are led to the idea of a surface where this result could not occur; of a surface, such as that of the paper or the table on which it is placed, wherein all straight lines, joining two points in the surface, and produced until they crossed, would not only have a point in common, but lie wholly in the surface. Such a surface is named *a plane surface*, or simply, *a plane*.

This new idea exhibits the preceding diagrams, not as models, but as mere pictures of the objects they represent; or, to speak more correctly,—it is from the com-

* Either of the angles BOC, AOD, AOB, DOC might be taken for the inclination of the lines AC and BD; which of them is to be chosen will be afterwards specified.

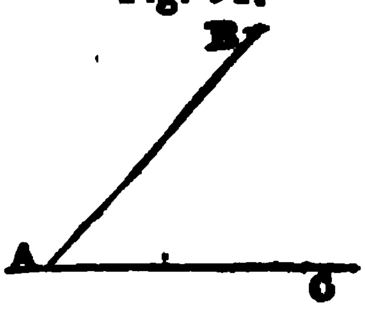
parison of four points that we are first led to reflect on the circumstances just mentioned, and to recognize among the ideas experience has furnished us with, this, which geometricians have named a plane surface.

Regarding the plane as a material lamina whose physical properties and thickness are neglected, we perceive that it is capable of being transported in space, and made to revolve about any line that it contains.

35. And from the last property it may be shown that a plane is determined in position when we know three points through which it passes, provided they are not in the same straight line.

For suppose the points were A, B and C; and imagine a plane passing through the indefinite right line AB, to commence revolving about that line, and towards the side where C is situated; after a sufficient revolution the plane will have attained C, and if its motion is continued, C will pass from lying on one side of the plane to lie on the other; whence it is obvious, that only one plane which passes through A and B can also pass through C.

Fig. 91.



And, in like manner, if we suppose two straight lines that have a common point, A, to be in a plane, they will also determine its position; since, by choosing other points B and C in each line, the theorem is reduced to the preceding.

36. When two planes meet, their intersection, or the points common to them, form a straight line; for, by the last article, three points not in a straight line will determine the plane wherein they lie, and cannot be common to two planes.

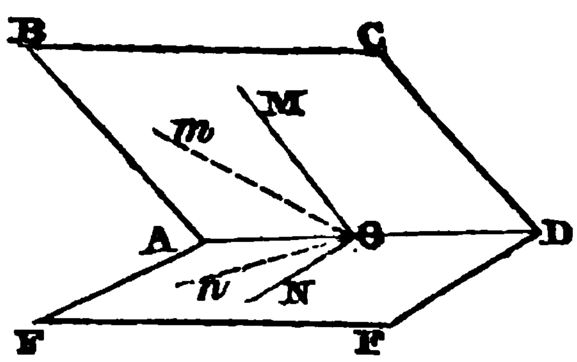
Chap. I. First principles of the science obtained from the relations of a finite number of points.

Art. 37. More perfect idea of an angle. Art. 38. Measure of the inclination of two planes.

37. The comparative direction of two planes affords the example of an angle different from those we have hitherto considered. But it should be remarked, that our notion of the inclination of lines, or the angle formed by them, was necessarily obscure, until we had obtained the idea of a plane, and demonstrated some of its elementary properties. The property, for example, that two straight lines having a point in common are in the same plane, will enable us to measure the inclination of such lines by the opening between them, or the portion they cut off from a plane infinitely extended.

38. And the inclination of planes might be measured in a similar way, and reduced to that of lines: for drawing from any point O, in their intersection, straight lines O m and O n, one in each plane; the inclination of these lines, provided the following conditions are fulfilled, will measure that of the planes.

Fig. 92.



1. After once the angles mOA and nOA have been assigned, they must in all cases be retained the same.

2. The same inclination must result wherever the point O is chosen.

3. When the inclination of the planes is varied, the angle m O n should vary in proportion.

These conditions are fulfilled, as will be shown hereafter, by drawing OM and ON at right angles to AO; and, accordingly, the angle formed by such perpendiculars has been taken to measure the inclination sought.

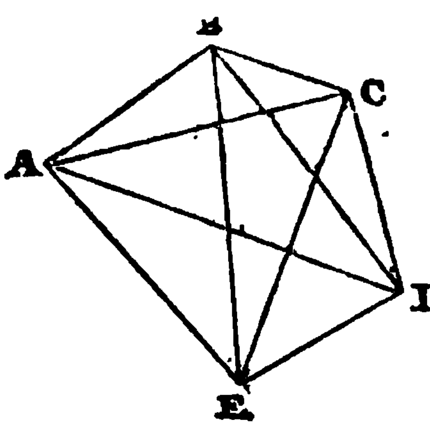
39. Extending our researches beyond the limited cases

discussed in the preceding articles, we still fall on combinations of the same relations: distances and simple directions result from arranging the points two and two, plane angles from a comparison of these directions, planes from an arrangement of the points three and three, solid angles from a comparison of the planes.

But immediately on entering this extended inquiry, we feel the want of a well arranged and uniform language, capable of expressing the data, and connecting, according to invariable laws, the results of each investigation. Nor is it easy to choose a notation that shall unite simplicity and an analogy to the ordinary language of algebra, with sufficient expression to serve as a convenient instrument of research.

A slight examination will convince us of this fact, and place in a strong light the difficulties to be reconciled. Turning our attention, for example, to the lines involved in the relations of many points, we immediately perceive the conflicting nature of the conditions required:—to preserve an analogy with the notation of algebra, each line should be expressed by a single letter, whilst to distinguish the lines, and call to mind their position, each symbol must comprehend many assertions. Thus considering A as a centre whence the lines AB, AC, AD, AE diverge; to distinguish them from others diverging at B, C, D, or E, their symbols should have reference to A; and, again, as AB, AC, AD, AE, all diverge from A, to distinguish these lines apart their symbols must have reference to the points referred to.

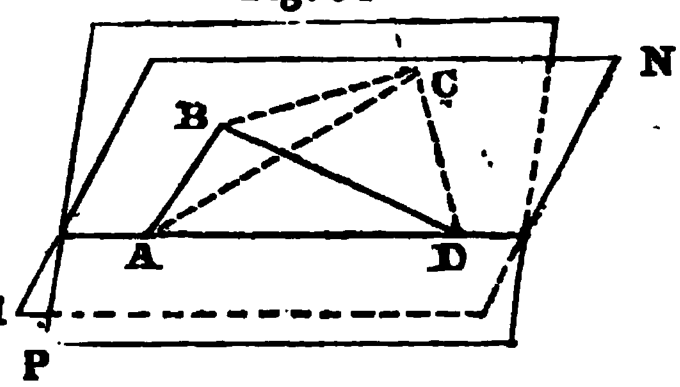
Fig. 93.



And similar remarks may be applied to the planes involved in the relations of many points, and to the signs used

to express them: but in this case every symbol must comprehend even more assertions than in the case preceding. For the sign of a plane should have relation to three points at once, since it requires that number to determine its position. Thus the symbol used to represent the plane MN, determined by the points ABD, should have reference to those points; as the symbol used to represent the plane PQ should have reference to the points A, C and D. But, unless it is otherwise expressed, the symbols will not be restricted to the positions of the planes, but represent certain portions of the latter—namely, in the first plane the symbol will denote the area ABD, and in the second the area CAD.

Fig. 94.



The symbol of an angle should express the lines or planes between which it is included; but as angles are usually preceded by the words cosine, sine, tangent, or other trigonometrical expressions that serve to separate the angle from the adjacent quantities, the language of algebra will be less departed from when angles are resented by combinations of more than one letter. And as it seems impossible to denote them by a single letter, and at the same time maintain a sufficient distinction between their quantities and the lines that include them, we must be contented with fulfilling in all other respects the conditions required.

The notation for lines, planes and angles, given in the annexed table, seems in a great measure to fulfil these conditions; and, accordingly, we shall employ it throughout the subject of our immediate inquiry—namely, the relations of a definite number of points.

Notation employed to express the relations of a definite number of points.

1. Points are expressed by the capital letters of the alphabet, A, B, C, &c.

2. The rank of these letters is denoted by their distance from A, thus,

0	1	2	3	4	5	6	7	8	9	
A	B	C	D	E	F	G	H	I	K,	&c.

Ex. The letter E is in the fourth rank, superior to F, and inferior to D.

3. A straight line drawn through a point is considered as *diverging from it.*

4. A straight line drawn through two points is considered as *diverging from* the point denoted by the superior letter. Ex. A line passing through the points B and E diverges from B.

5. Straight lines diverging from a point are denoted by the letter belonging to that point. Ex. A straight line diverging from A is denoted by a.

6. When many lines diverge from the same point they are distinguished apart by accents.

7. When a line diverging from one point passes through another, as many accents are placed over the letter denoting the line as there are units, wanting one, in the difference of the ranks occupied by those points. Ex. 1. A line passing through the points D and G is denoted by d''. Ex. 2. A line passing through the points A and F is denoted by a''''.

8. An angle formed by two lines is denoted by placing the letters expressing those lines in a parenthesis, and separating them by a comma. Ex. (a, b) denotes the angle formed by the lines a and b. But if the words arc, sin., tan., &c. precede the expression for the angle, the parenthesis and comma are omitted. Ex. The sine of the angle formed by the lines c and c' is written sin. $c\,c'$.

9. A plane passing through three points is denoted by the capital letter representing that one of the points which occupies the highest rank.

10. As many accents are placed *over* the letter representing a plane which passes through three points, as there are units, wanting one, in the difference of the ranks occupied by the two superior points.

11. And as many accents are placed *under* the letter denoting the plane as there are units, wanting one, in the difference of the ranks occupied by the two inferior points. Ex. A plane passing through the points A, E, H, is denoted by $A'''_{,,}$.

12. An angle formed by two planes is denoted by placing the letters expressing those planes in a parenthesis, and separating them by a comma. Ex. (A, B) denotes the inclination of the planes A and B. The notation admitting of a similar abbreviation to that in rule 8.

13. An angle formed by more than two planes A, B and C, is denoted in like manner by (ABC).

14. An angle formed by a straight line, a, and a plane, A, is also denoted by $(a\,A)$.

40. But though provided with a language capable of expressing these relations, we cannot hope to succeed in the analysis of them, until we have obtained some general principle that may serve to guide us in the investigation.

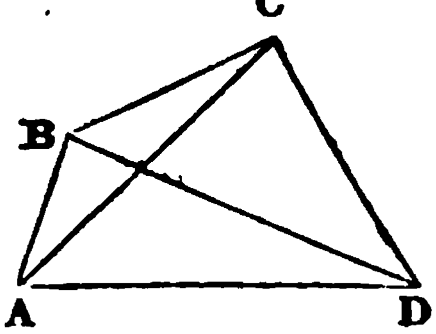
Such general laws, when pursued through the numerous ramifications they admit, assume a greater degree of simplicity than would be expected from the number of elements involved.

They may be arranged in a few classes, referring either to arrangements that can be made out of the required relations, or to simple elements, in terms of which all others should be expressed.

From the relations of three points, partially developed in the thirty-first, or from those of four in the succeeding article, it will appear that one of the simplest arrangements that can be made out of the relations of a definite number of points, is the *closed figure* or *polygon*; an idea that may be illustrated through the intervention of motion: for whatever may be the number of points, it is obviously possible to commence with any one, and passing through all the remainder, return without visiting any point twice, to the place where the route commenced; such a figure, returning back on itself, is called a closed figure, or polygon.

The relations of a greater number of points than three will admit of being arranged in more than one closed figure. Thus, the relations of the points A, B, C, D, may be analysed into four triangles, and three closed figures of four sides; the latter will be, ABCDA, ACBDA and ABDCA.

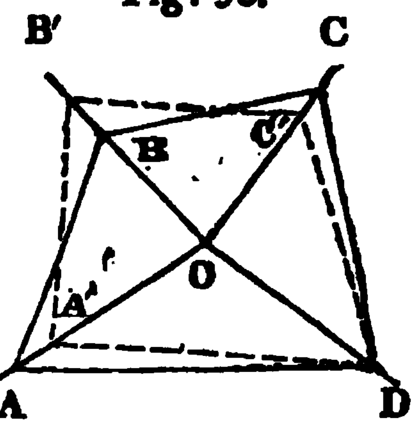
Fig. 95.



41. Leaving this subject to be further examined hereafter, let us proceed to a class of relations which at first appear distinct from those belonging to a definite number of points. I speak of the angles formed about a common vertex. From any point we may imagine indefinite straight lines diverging as from a centre—nor does there seem, on a casual observation, any connection between the angles included by these divergent lines, and the relations that form the subject of this chapter.

But a little reflection will lead us to perceive that it is always allowable to regard the relations of angles about a point as properties common to, and in some measure connecting an infinity of polygons.

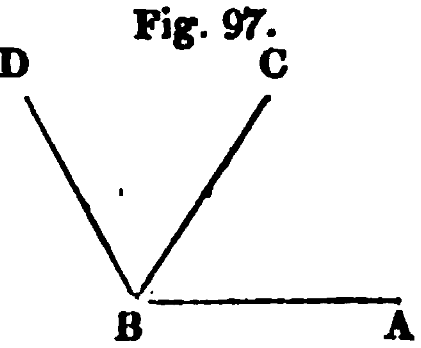
Fig. 96.



The angles about the point O, for example, may be regarded as belonging to an infinity of such figures as ABCD, A′B′C′D; whose sides, the last always excepted, are indefinite, and equal in number to the angles at O.

Commencing with the case where the divergent lines are in one plane, we remark in them the simple but important property of forming adjacent angles that are additive; the angles ABC, CBD, for example, if equal and adjacent, will form an angle ABD, which is double either of the parts. And as this property is obviously general, we must regard angles as "quantities," and therefore capable of comparison with a unit of their own kind.

Fig. 97.



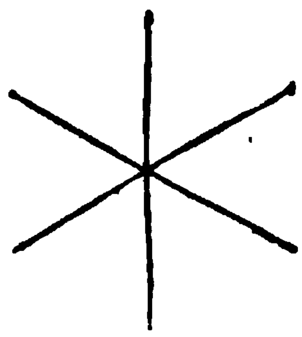
42. The choice of this unit seems too clearly indicated

by the nature of the quantities to admit of a doubt as to the angle that should be chosen.

In measuring most quantities the choice of a unit is arbitrary, but angles are so evidently portions of the space surrounding their vertex, and this space so manifestly the same in all cases, that we are forced to regard it, directly or indirectly, as the standard to which all angles should be referred.

Perhaps one of the simplest illustrations that can be given of the unit is to regard it as the sum of all those angles that admit of being drawn within one common plane and about one common vertex. For though the whole space, or, as we shall hereafter call it, the whole *plane-space* about the given vertex is, itself, an idea more simple than any of its parts, the latter will perhaps assist in obtaining a just conception of the former—being objects more readily presented to the eye.

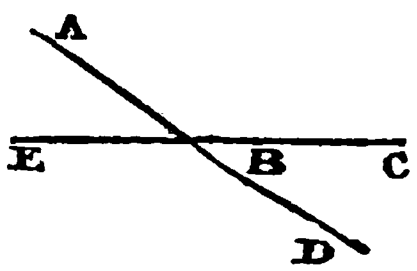
Fig. 98.



Assuming then the plane-space about a point as our unit of plane angles, it is abundantly evident, that wherever the point is chosen, the unit we have described, and consequently any portion of it, will remain the same.

And it will follow, as a corollary, that, when two straight lines meet, the vertical, or opposite angles, are equal: for since a straight line, produced indefinitely, divides plane space into two equal parts, the sum of the angles ABC and CBD, which compose the space on a side of AD, will be equal to the sum of the angles ABC, ABE, which compose the space on a side of EC; and taking from these

Fig. 99.



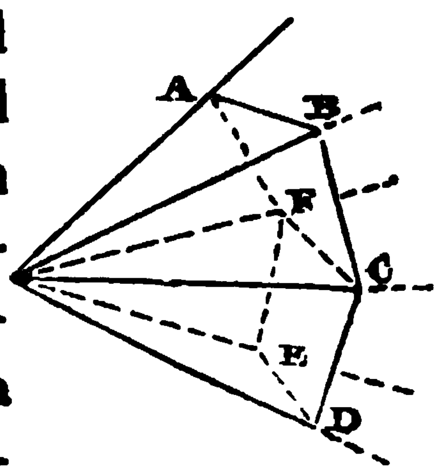
equal sums the common angle ABC, we shall have the angle ABC equal to the angle CBD.

43. The divisions of this unit that most frequently occur, are the half, and the fourth part.

The latter division is named the *right angle*, and the difference between it and any other angle is called the *complement* of the latter, as the difference between the half unit and any angle is called its supplement. Either division, for reasons we shall hereafter explain, has very generally been considered as the unit of angles.

44. When the lines diverging from a point do not lie in one plane, the angles they form are no longer additional: but among the possible series of the plane angles it is obviously possible to choose such an arrangement, that, to illustrate our meaning through the intervention of motion, we might commence with any angle, and passing over all the remainder, and without visiting any one twice, return to the angle the route commenced at: plane angles so arranged, involving a definite portion of the whole space about their vertex, are said to inclose a *solid angle*.

Fig. 100.



45. But supposing a plane, as ABCF, or CDEF, opposite to each of the solid angles about a point; the sum of all these planes, it is evident, will form the surface of a *closed solid*, or *polyedron*. An idea, perhaps, more immediately derived from the relations of a definite number of points; since by an-

alyzing the latter, as before mentioned, into combinations of threes, art. 39, and passing planes through each of these combinations, we form an assemblage of planes capable of being analysed into many closed solids.

Uniting both views, we shall conclude from arguments analogous to those in art. 41, that solid angles having a common vertex, produce relations common to an infinity of polyedrons. The solid angles at O, for example, may be regarded as relations—not only of the closed solid OABDEO, but of any closed solid that could be formed by planes opposite to O, and intercepted between the faces of the solid angles there.

46. And, art. 44, bearing in mind that angles of the kind last mentioned are definite portions of the space about their vertex, we immediately perceive this space to form a unit of comparison for such angles, and can apply to it, with proper modifications, the remarks made concerning the infinite plane assumed as the unit of plane angles; observing however that the solid right angle is, not the fourth, but the eighth, part of its unit.

From a casual view of the definition in art. 44 it would appear that a solid angle could not be inclosed by less than three planes, but if we turn to art. 9 we shall observe the solid enclosed by two planes, mentioned in that article, which solid is the true measure of their inclination, to be equally a definite portion of the whole of space, and therefore of the same nature with the solid angles about a point.

Nor will it be difficult to show the relation, or, rather the identity of these two species of angles.

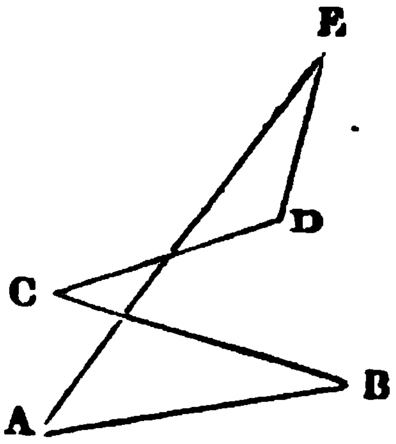
But to attain this end we must first demonstrate that

…the su…
to arrange and …
pushing our inquir… …at all the rela-
truths presenting thems… …btained from
necessary to follow methods … …gures and
lar problem in view—methods ded… …ple, and
from general principles, and working ac… …es to-
rules.

That such principles cannot be very remote is … from the limited number of elements admitted—the dis-tance between two points—the angle made by two lines—are the *simple* elements out of which all the relations of a definite number of points may be compounded.

Now whilst the distance of the point B from A, C from B, D from C, E from D, are the only data; or, more generally, whilst the system can be passed over in such an order, that, for each point, the only given relation is its distance from the point preceding: no conclusion can be drawn respecting the remaining relations.

Fig. 101.



But connect E with A—the first point with the last—and the remaining relations will be subject to a condition!—the angle E, for example, is made dependent on those at D, at C, and at B.

But to make this connection is to include all the given

lines in a closed figure: and had E been connected with any other point in the system, with C, for example, similar results would have followed, the lines between C and E would have formed a closed figure and their angles been subjected to a condition.

It is true that when the data include angles, a condition may be involved among the angles remaining, the inclinations, for example, of ED and CD to AB, when the lines are in one plane, will involve the inclination of ED, to CD; but as the conditions in this case, as we shall show hereafter, are the same with those of angles having a common vertex, we are led to this remarkable conclusion, that, every proposition relating to a definite number of points may be deduced from the properties of one or more closed figures. In practice it will be often convenient to analyze by polyedrons instead of polygons, but this fact does not restrict the generality of the preceding remark, since the latter species of analysis will be shown to include the former.

PRELIMINARY REFLECTIONS TO SECTIONS III. AND IV.

From Sections I. and II., we learn that all the relations of a finite number of points can be obtained from those of closed figures and solids:—such figures and solids can be decomposed into others more simple, and again compounded by putting these simple figures together.

INQUIRIES SUGGESTED BY THESE REFLECTIONS.

Can all closed figures be decomposed until their parts are alike:—what simple figure, or type, results from this decomposition:—in what manner are these types to be placed side by side so as to compose any given figure?

Can the same be done for closed solids—in what manner are we to discover the relations of these simple figures, or types of comparison?

SECTION III.

RELATIONS OF THE TYPE TO WHICH CLOSED FIGURES ARE COMPARED.

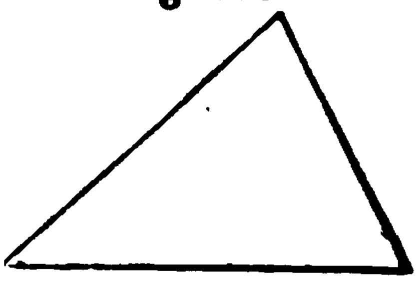
Relations of three points resumed—symmetry of figures—principle of elementary figures—the triangle which has one right angle assumed as a type of comparison for other triangles—the relations of this type deduced from the principle of superposition—relations of the type—comparison of other triangles with the type performed by superposition—further remarks on the connection between the parts of the type—various principles on which the science may be founded—infinity of space a notion essential to geometry—principle of homogeneity—symmetry of figures.

48. The form of our analysis was in some measure measure determined in the preceding section, where all propositions respecting a definite number of points were shown to resolve themselves in the relations of closed figures. But several general principles have yet to be explained before the application of this analysis can be advantageously developed; and with this view it will be

necessary to resume for a moment the inquiry into the relations of three points.

We have already found, art. 31, the relations in question to form the closed figure called a triangle, and to include three distances and three angles. And as we arrived at the notion of this triangle without selecting any one point in a manner that would cause it to be involved in a different way from the rest; it follows that whatever conclusion is obtained by selecting a particular side or a particular angle, provided it is true for every form of the figure, a similar conclusion ought to follow from employing either of the other sides, or either of the other angles.

Fig. 102.



Thus, for example, if we prove a relation to exist between AC and the opposite angle B, fig. 103, and also prove this relation to hold for every form of the triangle; it must then necessarily follow that the same relation exists between the side BC, and the angle A.

Fig. 103.

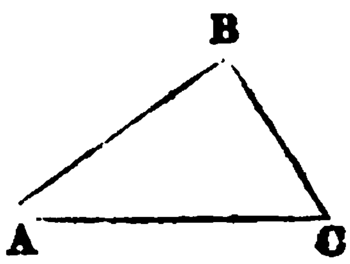
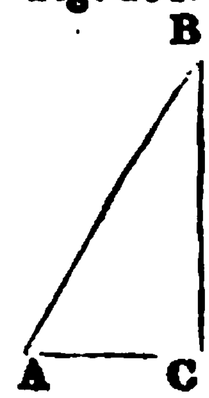


Fig. 104.



For among the possible forms of the triangle we might choose one, fig. 103, wherein the sides and angles were interchanged—the side AC becoming equal to the side BC of the primitive figure.

Now as the result is true of the side AC and the angle B, whatever may be the form of the triangle, it must be true of that side and angle in fig. 103; but AC and B in the latter figure correspond with, and are identical to, BC and A in fig. 103, and whatever is true of the one must be equally true of the other.

Hence, unless we suppose, which involves an evident absurdity, the properties of figures to change with their change of position or place, we must conclude that whatever is fully proved of one part must be true of all the parts that correspond to it; a result which brings the inquiry within a narrower compass, enabling us to extend the relations demonstrated for one particular side of a triangle to its other sides.

48. The inquiry is further reduced by employing the principle of *symmetry*—a fundamental property of geometrical figures, and which asserts their form to depend upon the ratios, and not on the actual magnitude of the sides.

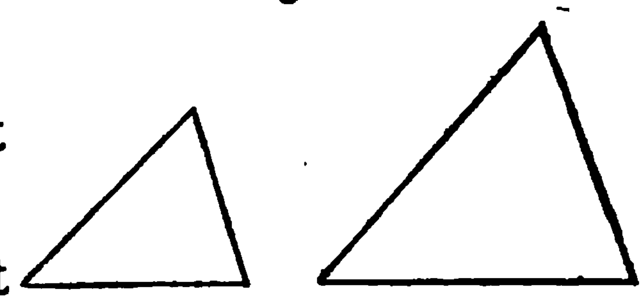
The principle in question can be readily deduced from the proposition that innumerable figures exist having the same form but different magnitudes; and, accordingly, when our analysis is sufficiently advanced, we shall prove the mutual dependence of these theorems.

Their truth, however, though not yet established, has been gathered from observation in Part I. and may serve as a guide for our future researches, suggesting to us an important reflection; for since number expresses the ratios of lines, and not their actual lengths, we shall conclude that if the propositions above adverted to are just, the relations of geometry can be reduced to those of number.

The course of our inquiry thus determined, we shall advance with greater certainty by restricting it to the more elementary of the two propositions, and by confining that proposition to the simplest figures.

Following this route, and reasoning only on triangles,

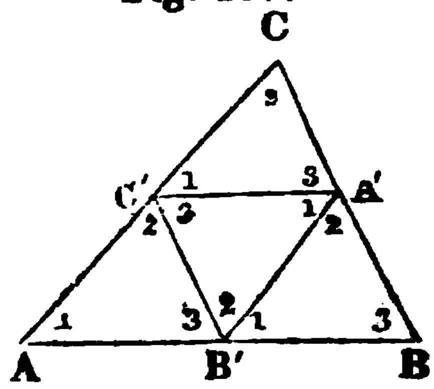
Fig. 104.



it seems impossible to deny that with respect to such figures the form is independent of the magnitude.

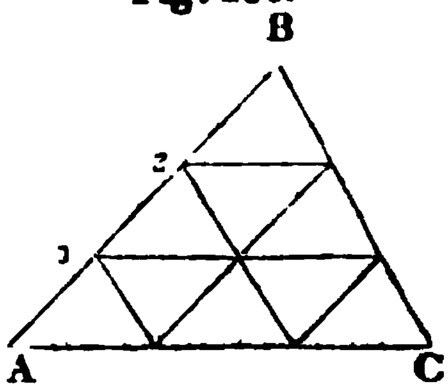
But as principles ought not to be assumed that by analysis can be reduced to others more simple, it becomes incumbent on us to examine this supposed property of triangles, and trace, if possible, the ultimate relations on which it depends.

Fig. 105.



Taking then a triangle AB′C′, and disposing other equal triangles about it, arranging as in fig. 105; a mere inspection will show that if the sums of the angles at the points A′, B′ and C′ were each equal to two right angles, the lines AB, BC, CA, would be straight lines; and the figure ABC a triangle having its angles equal to those of the triangle AB′C′.

Fig. 106.



And proceeding to a similar construction, but where a greater number of equal triangles are arranged together, we are led to generalize this result, and to conclude that if the three angles of a triangle were always equal to two right angles, innumerable triangles could be constructed, varying in magnitude, but retaining the same form.

The sum of the three angles of a triangle is obviously therefore our next inquiry; but this problem will be greatly facilitated by the steps we have already made; since it follows from those steps that, if we can show the three angles of any one triangle to be equal to two right angles, the same equality will subsist for all triangles de-

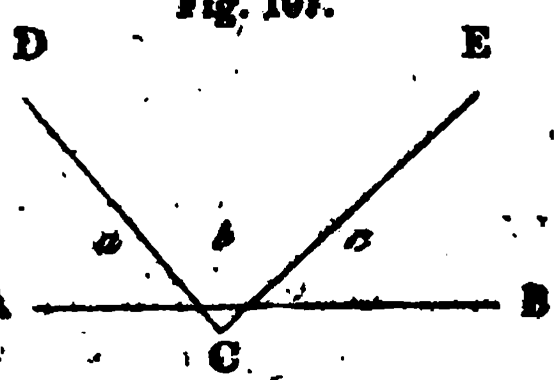
rived, according to the preceding construction from this one.

49. And reversing our steps, and decomposing triangles into parts, each resembling the original in form, but getting less and less; and pushing the decomposition until the parts, which we shall then call *elements*, become less than any assignable triangle; it will still follow that we shall have established the property in question for the whole triangle, whenever we establish it for one of the parts.

But figures so small will be little distinguishable from points, and the angles formed by their sides, indefinitely produced, will be nearly the same as those formed by lines about a point.

The lines AB, CD, CE, for example, that enclose a small triangle at C, are separated by openings *a*, *b*, *c*, that are nearly equal to the angles of the triangle: two of these openings namely *a* and *c*, are identical with angles of the triangle, and the third, *b*, which forms a space infinitely extended, differs from the opening we call the angle C, merely by the small space included in the triangle.

Fig. 107.

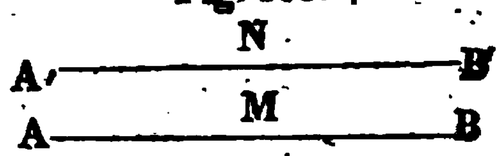


This last, by bringing AB nearer to C, may be rendered as small as we please: and thus a triangle can always be assigned whose angles shall differ from *a*, *b*, *c*, and, consequently, the sum of whose angles shall differ from two right angles by less than any assignable quantity. Some difference between the results appears, it is true, always to remain; but if we examine more attentively the idea that we are able to form of infinite space, we shall find the difference in question merely apparent, and shall perceive the sum of the three angles of a triangle to

be rigidly equal to two right angles. The former sum, we have seen, differs from the latter, or from the half of plane space, merely by the finite area included in the triangle; and with regard to this difference we may reason as follows:

Drawing, at pleasure, a straight line AB; the infinite plane spaces on either side are each equal to two right angles, and to each other; and as the same may be said with regard to a second straight line A′B′, so drawn as not to meet the first, however far produced, the plane space on either side of AB, which space we shall denote by P, will be equal to the space N, on either side of A′B′; and we have the equation P = N. Now the space P is composed of two parts, the part M, that lies between the lines, and the part N, that lies beyond A′B′; and expressing this algebraically, we have P = M + N. But equating these equal values of P, we also obtain N = M + N; and as this result is strictly deduced, it proves, however obscure the fact may seem, that either all our notions concerning angles are erroneous, or the addition of such a space as M, which is of finite breadth, will not alter a space N, that is infinite in two directions.

Fig. 108.



But admitting this conclusion, which we have seen is rigidly demonstrated, it will also follow, and for a stronger reason, that the sum of the three infinite spaces, *a*, *b* and *c*, fig. 107, is not altered by the addition of the triangle at C; or, in other words, that the three angles of a triangle, whatever may be its magnitude, are equal to two right angles.* To render this result more useful, we will employ it to demonstrate these additional properties.

1. If two triangles have a side and any two angles of the one, equal to a side and any two angles of the other, the triangles will be equal.

For the three angles of each triangle are equal to two

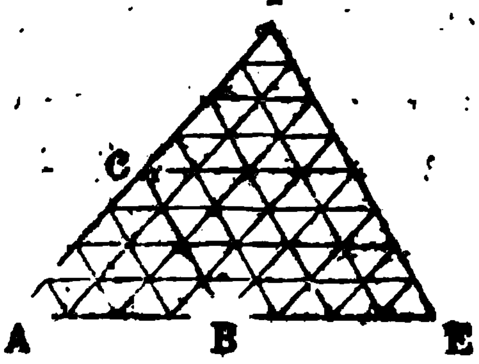
* Note 1.

right angles; and if from this sum we subtract the two given angles of the first triangle, and from the same sum the two given angles of the second triangle, the remainders, or third angles, will be equal. And hence, superposing the figures, and adjusting the sides that are equal, the remaining sides of the first will take the same directions as the remaining sides of the second, and the several parts of the two figures, falling on each other, will agree, and the two triangles will be equal in all respects.

2. If two triangles have their angles equal, the ratios of the corresponding sides, that is, of the sides about the equal angles, will also be equal, and the figures will be *similar* or *symmetrical*.

For suppose ABC and AEF two triangles, the sides of which, AC and AF have some common ratio. Since, by the process already described, and by taking for the side of the elementary triangle a common measure of AC and AF, we can decompose the figures into elementary parts, whose sides shall be as small as we please; it follows, that $\frac{AC}{AB} = \frac{AF}{AE}$.

Fig. 109.



This proof, it will be observed, is only given of triangles, the sides of which are measured by some common unit; but since it has been proven before, in the elements of algebra, that relations which extend to all quantities that can be measured by the successive additions of an indefinite unit, are, in fact, general, the result in question is true of all triangles that have their angles equal.*

50. The principles here established are sufficient to prove that all the linear relations of three points, and we shall afterwards render the remark general, can be reduced to those of number.

* Note 2.

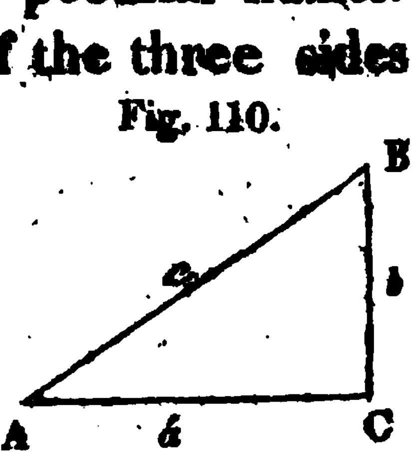
But this fact, important as it undoubtedly is, forms but a single step of our inquiry; and notwithstanding that it restricts the latter, at least as far as this section is concerned, to the relations existing between the angles and the ratios of the sides of a triangle, the subject is still of great extent, and in order to pursue it with advantage, some further reductions must yet be employed. With a view to these reductions, let us commence the investigation with one of the least complicated forms which triangles admit; and by a comparison of other such figures with this simple type, observe the changes that must be made in the conclusions to adapt them to every species of triangle.

Some of the simplest forms of the figures alluded to do not admit of a ready comparison with other triangles, and paying the due attention to this requisite, we shall find the species of triangle which has one of the angles a right angle, to answer the purpose of a type better than any other form of that figure.

But as the principles which have already been developed permit us to compare together right angled triangles of any magnitude, and prove the form to depend altogether on the ratios of the sides, it becomes an object of importance to distinguish these ratios by peculiar names.

The ratios that can be formed out of the three sides are six in number, but three of these, it is evident, are merely the remaining three inverted, that is, with the numerators and denominators interchanged. The names imposed on these ratios will be understood by the following table.

Fig. 110.



$\frac{\text{Side adj. to A}}{\text{hypothenuse}}$ or $\frac{a}{b}$ is named cosine of A

$\frac{\text{Side opp. to A}}{\text{hypothenuse}}$ or $\frac{b}{c}$ is named sine of A

$\frac{\text{Side opp. to A}}{\text{side adj. to A}}$ or $\frac{b}{a'}$ is named tangent of A

The reciprocal of the cosine, or $\frac{c}{a'}$, is named secant of A

The reciprocal of the sine, or $\frac{c}{b}$, is named cosecant of A

The reciprocal of the tangent, or $\frac{a'}{b}$, is named cotangent of A.

The angle B has also its sine, cosine, &c. and since a mere displacement of the paper would cause the letter B to occupy the place of A, the ratio which is the sine of A must be the cosine of B; and, in like manner, the tangent B is the cotangent of A, and the secant of B the cosecant of A.

50. 1. Having named the several parts of the type, our next step is to discover their relations, a process of a kind different from any that has yet engaged our attention; but that proceeds, in like manner, by the arrangement and comparison of figures.

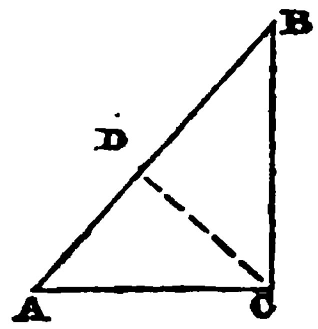
The relations already noticed in the ratios, namely, that some are reciprocals of others, is an immediate result of our own convictions, and does not require demonstration; but if we seek to discover the relations that are peculiar to the figure under consideration, the want of a principle on which to found our inquiries is immediately felt, nor can we proceed in the investigation until such an instrument is provided.

The method of superposition—a method, as we have formerly remarked, equally common to the sciences and arts, supplies us with this instrument, and enables us to

compare figures by proving certain of the parts into which they are divided, identical in each.

The choice of the right angled triangle as a type of comparison will direct us to divide other figures into right angled triangles, and thus to analyze the sides and angles of the parts of the integral figure into parts that are found in right angled triangles.

Fig. 111.



This application of the principle of superposition regards all figures as made up, or covered by, right angled triangles; and is as applicable to the type itself as to any other variety of form.

Thus, selecting two right angled triangles, ACD, CBD, that together are identical with the right angled triangle ACB, we divide the sides and angles of the latter, or integral figure, into parts that are found in one or other of the two smaller triangles.* But

$$a=a''+b'$$

And

$$\frac{a''}{a'}=\cos. A=\frac{a'}{a}$$

$$\frac{b'}{b}=\cos. B=\frac{b}{a}$$

From the two last equations we derive

$$a''=\frac{a'^2}{a}, \quad b'=\frac{b^2}{a};$$

and, substituting in the first, there arises

$$a^2=a'^2+b^2.$$

* $AB=a$, $AC=a'$, $AD=a''$, $BC=b$, $BD=b'$, $CD=c$. See notation, p. 109.

Or, calling the side AB, opposite to the right angle, the hypothenuse,

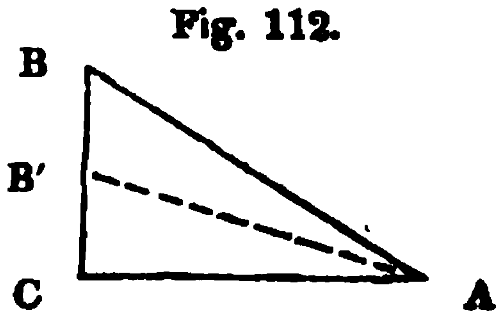
The square of the hypothenuse is equal to the sums of the squares of the other two sides.

A property which is characteristic of the type.

50. 2. Another property of the latter will immediately follow from the principle already demonstrated, that in every triangle the sum of the three angles is equal to two right angles. For since the angle at C is a right angle, the sum of the remaining angles will be together equal to a right angle; and "the two acute angles will be complementary to each other."

50. 3. A third property relates to the connection between the angles and the ratios of the sides; and may be demonstrated by drawing a line from one of the acute angles, A, to a point B′ in the opposite side, fig. 112.

Fig. 112.



By this construction we form a second right angled triangle, B′AC, that has the side AC in common with the first, but the side BC, and the angle at A, less.

The ratio of the opposite side to that adjacent, or the tangent of A, will, therefore, be least in the triangle which has the least angle at A. And as this result is true whatever may be the magnitude of A, or the position of B′, we conclude, generally, that with respect to all "acute angles the tangent increases and decreases with the angle."

By assuming B′ very near to C, the angle A may be made to approach as nearly as we please to zero, and consequently the angle B to a right angle, art. 49; but

as B′C diminishes, the ratio $\frac{AC}{B'C}$, or the tangent of B, will increase, and as the one approaches to zero, the other will approach to infinity.

From all which, we conclude, first, that "a right angled triangle may have its acute angles of any value between zero and a right angle;" and, secondly, that "as any angle increases from zero to $\frac{1}{4}$, its tangent increases from zero to infinity."

The results with respect to the sine are nearly similar; for as the square of the hypothenuse is equal to the sums of the squares of the other two sides, art. 50—1, it will follow, that if one of the sides is diminished, the other remaining the same, the square of the hypothenuse will diminish less rapidly than the square of the decreasing side, and the ratio of that side to the hypothenuse will diminish with the former.

This ratio becomes zero with the angle at A, but it does not become infinite when the latter is a right angle. In fact, since $AB'^2 = AC^2 + B'C^2$, fig. 112, if B′C approaches zero, AB′ approaches AC, and the ratio of AC to AB′, which is the sine of B′, approaches to unity.

From this reasoning we conclude, that "as an angle increases from zero to $\frac{1}{4}$, its sine increases from zero to unity."

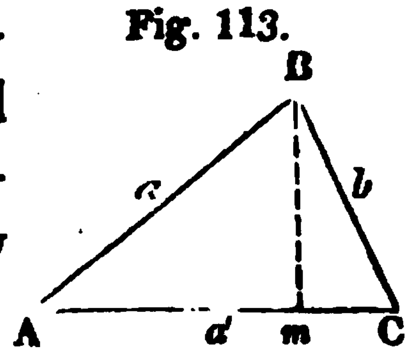
The cosine of an angle follows a contrary rule, for the sine of A is the cosine of B, and the latter angle decreases with the increase of the former,—art. 50—2. The cosine of B, therefore, increasing with the sine of A, and consequently, by the preceding part of this article, with the angle A itself, increases a sB diminishes.

And we conclude, that, "as an angle increases from zero to $\frac{1}{4}$ its cosine diminishes from unity to zero.

The relations between the angles of the type and the ratios of its sides may be pursued further, but before we enter into this inquiry it will be convenient to investigate a process by which the relations of every species of triangle may be reduced to those of the known standard.

51. Comparing the triangle ABC with the given type, we observe, that the right angled triangles, ABm, and BCm, may be so placed as conjointly to coincide with the triangle ABC; and consequently, that, by what has preceded we shall have

Fig. 113.



$$\frac{Am}{a} = \cos.\, aa'$$

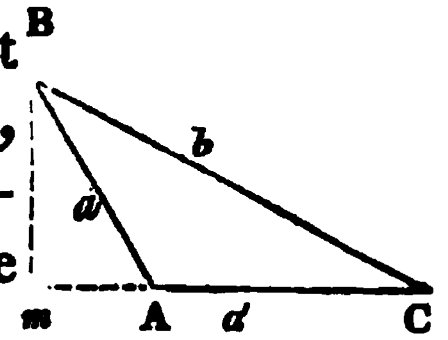
$$\frac{Cm}{b} = \cos.\, ba'$$

or clearing the equations of fractions and adding them together,

$$a' = a \cos.\, aa' + b \cos.\, ba'.$$

When the given points are disposed as in fig. 114 the point m no longer falls within the triangle: and we also meet with a doubt not suggested by the former figure, namely, whether to measure the inclination of the lines BA and AC by the angle BAC or its supplement.

Fig. 114.



To solve this doubt we must have recourse to the preceding case, and as we there made use of the inter-

nal angles of the triangle, the unity of method which analysis requires will direct us to measure the inclination in question by the obtuse angle BAC.

The true reason, however, for choosing this measurement is the power it gives of extending to the present case the formula deduced from the preceding. For observing that a right angled triangle cannot contain an obtuse angle, we perceive, art. 50, that such angles will have neither sine nor cosine unless a further convention is made respecting those terms. But agreeing that an angle has the same cosine as its supplement but with a contrary algebraic sign prefixed to it, we have

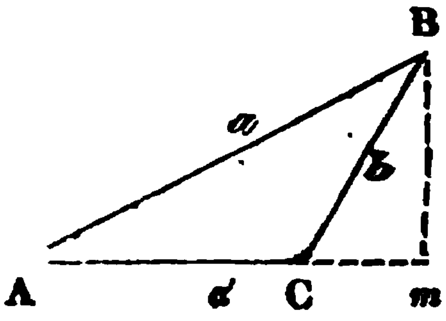
$$Cm = b \cos. ba'$$
$$Am = -a \cos. aa'$$

and, subtracting these equations, there still results

$$a' = a \cos. aa' + b \cos. ba'.$$

Lastly, when the given points are so disposed that the angle C is obtuse; similar results will follow from similar reasoning: and as the case where one of the angles is right, is merely a limit between those where it is obtuse and where acute, we conclude that the expression obtained for the side a' extends to all triangles, and is independent of the position of the points whose relations are sought: moreover, observing how the several quantities that enter this equation are placed in the triangle with respect to the selected side a', we have only to substitute for them other letters similarly situated with respect to another side, in order to deduce a value of this last.

Fig. 115.



By such substitutions we obtain the three equations

$$\begin{aligned} a &= b \cos. ba + a' \cos. a'a \\ b &= a' \cos. a'b + a \cos. ab \quad . \quad . \quad . \quad . \quad (1) \\ a' &= a \cos. aa' + b \cos. ba' \end{aligned}$$

which, involving all the properties of the triangle, reduce the chief subject of this section to a problem purely algebraic. And as the reader is supposed to have previously studied the elementary rules of algebra, all results which flow from the application of those rules to the equations (1) may be justly regarded as already established. The great importance of these results will, indeed, render it *convenient* to give this development, but the place it shall occupy may be postponed at pleasure; and will not occur in this investigation until Part III.; where it will be shown that of the six elements which constitute a triangle, any three, except the three angles, are sufficient to determine the remainder.

52. All that is now wanting to complete the theory of triangles is the investigation we proposed deferring, and which has for its object to determine the relations between an angle and its cosine, sine, &c. Until this investigation is completed the equations (1) can only establish the connections between the sides of a triangle and those of the right angled triangles that are together equivalent to it, that is, which exactly fill the same space.

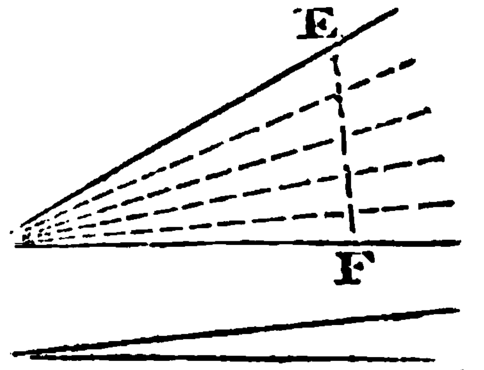
Even this imperfect analysis, however, is of wide application, and should we fail in the further inquiry, so often mentioned as deferred, its place, as far as all practical purposes are concerned, might be supplied by calling in the assistance of the artist.

Providing ourselves with a large and smooth sheet of some soft metal, we might commence this practical in-

vestigation by tracing on the surface an extensive series of angles, each increased beyond that preceding it by some constant and small angle BAC.

From this series an equal number of right angled triangles might be obtained by letting fall, in each of the angles, a perpendicular EF, on one of the sides, from D a point E taken at pleasure in the other.

Fig. 116.



When, finally, our problem would be resolved by measuring the sides DE, EF, FD, and arranging their ratios in a table under the names of cosine, sine, tangent, &c.

Such a table as this will be found the only practicable way of preserving for immediate use the results of our researches on the right angled triangle: other methods of calculation, however, must be had recourse to: appeals to the testimony of the senses can only be permitted when all other modes of research have failed; and in place of the graphic investigation above described, we must substitute a purely mental process, capable of developing the numbers in question, not only with greater accuracy than could be attained by the most refined processes of art, but with any degree of accuracy required. A method of this kind will be explained in the section already referred to; and, in the mean time, let us pursue our reflections on the relations of points and the methods of comparing their relations with those of the standard figure.

53. The theory of triangles established in the pre-

ceding articles, has been chiefly deduced from the principle of elementary figures, or of figures possessing the same form as that of which they are parts, and capable at the same time of being reduced to dimensions less than any assignable limits.

But in whatever manner the foundations of our science are established, it is essentially necessary that we should previously deduce the relations of the right angled triangle, or of some other simple standard; and in this preliminary operation much difficulty was experienced by those who first treated the subject; until, with enlarged notions in every branch of mathematics, more just ideas arose concerning these elementary truths; and the failure of so many eminent mathematicians to establish their science on the single principle of superposition, was justly regarded as indicating the existence of other equally fundamental principles.

54. The idea of superposition, whilst it supposes a certain knowledge of the properties of space, does not include the notion of its infinity, a notion essential to our subject, since we cannot form the idea of a simple direction without further supposing the possibility of producing that direction infinitely on either hand, nor obtain just notions of an angle without supposing it a definite portion of that boundless plane in which the sides that contain the angle are found.

The idea of infinity is implicitly contained in the principle of elementary figures, as well as in that of two other principles, that, taken as original truths, would equally serve to support the fundamental propositions of geometry.

Chap. I. First principles of the science obtained from the relations of a finite number of points.

Art. 55. Symmetry of figures. Art. 56. Principle of homogeneity.

55. The symmetry of figures, the first of these two principles, has already been explained and deduced with respect to triangles, from the course of reasoning adopted in art. 48.

56. The other principle alluded to, the principle of *homogeneity*, is of equal importance, and will require a few words of explanation. Taken in its most general sense, it asserts the self-evident proposition that heterogeneous quantities can only be compared by the ratios they bear their units; but when the principle is examined in detail, it becomes necessary for us to define the quantities that are heterogeneous. Thus, among other examples, lines, whilst they are asserted to be homogeneous to lines, are said to be heterogeneous to angles; by which is meant, that we cannot obtain the quantity of an angle by taking any number of times the quantity of a straight line.

Legendre, from the principle of homogeneity, has undertaken to derive the theory of triangles, but whilst the method he pursues is excellently adapted to exhibit the relations of geometry to those who are already acquainted with that science, it fails as an elementary process of investigation; requiring us to assume, what is by no means obvious, that all the relations of geometry, or in other words, of position, can be reduced to relations of number.

SECTION IV.

RELATIONS OF THE TYPE TO WHICH CLOSED SOLIDS ARE COMPARED.

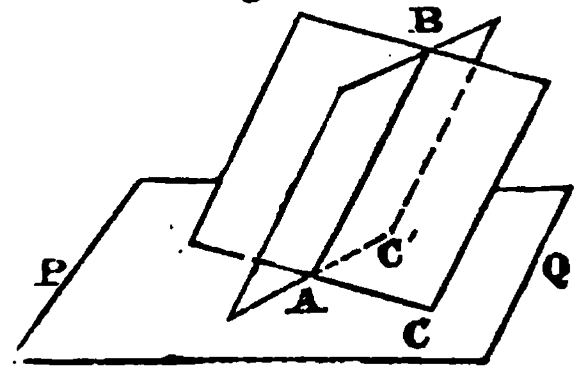
Inclinations of lines with planes—the triangular pyramid with one solid right angle—the rectangular pyramid—the rectangular pyramid chosen as the type of closed solids—relations of the type—angle which measures the inclination of a line to a plane—particular cases of such angles—measure chosen for the inclination of planes shown to fulfil the necessary conditions—comparison of closed solids with their type—vertical solid angles are equal—comparison of solid angles contained by two planes with those contained by three or four.

57. From the species of angles we have hitherto considered, we naturally pass to the inclinations of lines with planes; but a slight attention to the subject will convince us that to measure angles of so distinct a nature, some new convention becomes necessary.

To render this apparent, let us assume the line BA inclined to the plane PQ, and pass through BA a plane

BAC, such that its intersection with PQ shall be the line AC; the inclination of the line and plane might evidently be measured by BAC: but other measures may also be obtained by a similar construction. Thus passing through BA another plane BAC′ whose intersection is AC′, the inclination in question may be measured by BAC′; and since BAC and BAC′ are not equal, it becomes necessary to decide by agreement, which angle, out of the infinite number so formed, shall represent the inclination of the line and plane.

Fig. 117.



This choice may in some respects be considered as an arbitrary one, yet it will be proper to examine the angles we choose among, and select that best adapted to our purpose.

Such an inquiry, involving the relations of planes that pass through a common point, supposes an acquaintance with solid angles: and as we have ascertained that solid angles are best investigated by means of closed solids, it will be expedient, before proceeding further, to examine the simplest cases such relations offer.

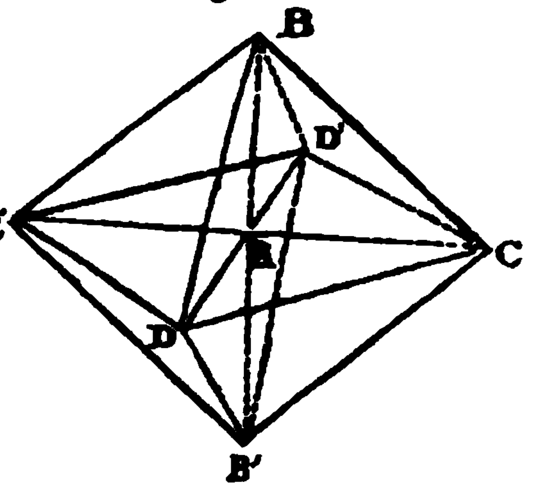
The principle that guided us through the analysis of plane figures, will be equally useful in this instance; and accordingly, we ought to seek for a type that shall fill, with respect to solids, the place which is occupied, among the figures alluded to, by the right angled triangle.

58. But the simplest solids are found among the relations of four points: and, taken three and three, the

combinations of four points determine the positions of as many planes, and thus give rise to elementary solids that are classed under the general appellation of triangular pyramids. Agreeing, in many respects, with the triangle of plane geometry, they also admit of similar cases; among which, from its analogy to the right angled triangle, we cannot fail to remark the *triangular pyramid having one right angle.*

The existence and nature of this solid may be readily shown: for drawing BB′ at right angles to a given straight line CC′, we may apply to the latter a triangle CC′D, composed of the two right angled triangles CAD and C′AD: and causing this triangle to revolve round CC′ it may be made to assume such a position that the angle BAD shall be a right angle. Whence assuming B, C, D, B′, &c. at equal distances from A, and in the lines BA, CA and DA produced, the planes determined by these points will form eight solids, each inclosed by four triangular faces; and as the triangles at A have each three parts the same, namely, two sides and an included angle, by art. 51 they are equal; and, consequently, their bases are equal; and an equality exists between all the corresponding faces of the eight triangular pyramids.

Fig. 118.



But solids so constructed, admitting of mutual superposition, are identical; and thus, the angle at A, in any one of the pyramids, being the eighth part of the space about A, and equal in all respects to the remaining angles about that point, is, by art. 46, a solid right angle.

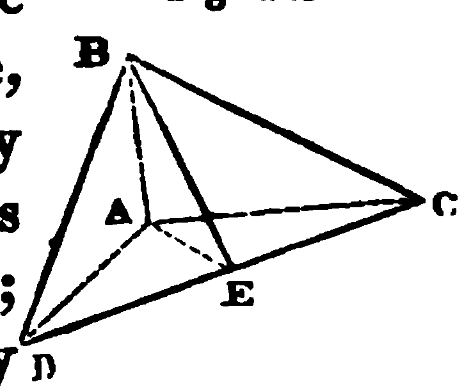
Now this angle is not dependent on the values of AB, AC and AD, but will remain the same whatever is the

relation of these lines. And thus, constructing a solid right angle as above, and in the intersections of the planes bounding this angle, taking points, B, C, D, at pleasure, and finally, passing a plane through BCD, we form the pyramid in question.

Such a solid appears to have the relation to triangular pyramids which the right angled triangle has to other triangles; but, notwithstanding this apparent resemblance, the triangular pyramid with one solid right angle, is not well adapted to serve as a figure whence others may be compounded, and, in this respect, is greatly surpassed by a form resulting from its decomposition.

59. To obtain this form, draw BE perpendicular to CD, and pass a plane through A, B and E, either of the solids ABCE or ABDE will then be the pyramid we seek.

Fig. 119.



From the construction each of the angles BAC and BEC is a right angle, and as BAE and AEC will presently appear to be so likewise, the solid has all its faces right angled triangles; and, in allusion to this property, may be conveniently denominated *the rectangular pyramid.*

60. The nature of this solid renders it readily susceptible of being compounded with others of its own kind, and as by a proper system of juxtaposition we may form any closed solid from it, we shall select the rectangular pyramid as a standard, or type of comparison for all such solids.

Sect. IV. Relations of the type to which closed solids are compared.
Art. 61. Relations of the type.

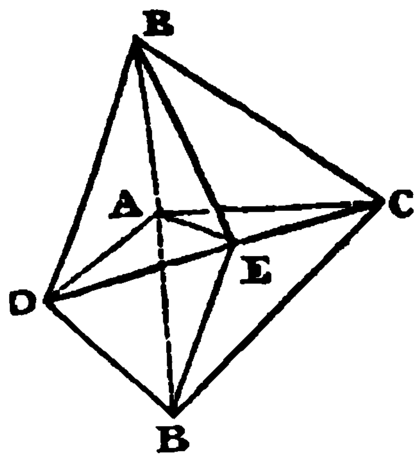
61. This choice completed, it will be proper that we should proceed, as in the former case, to investigate the chief properties of the type.

The analysis may be performed by means of the relations discovered among the parts of the right angled triangle, sect. 3, but as considerations flowing from the construction of the type greatly facilitate the process, we shall avail ourselves of their assistance.

The first object is to prove the characteristic property that all the faces of the solid are right angled triangles.

The faces A and $B_{,}$* are right angled by construction; and thence we readily prove the same thing of the face $A_{,,}$; for referring to art. 58, it will be seen that a solid identical with ABCD may be placed below it, and in such a manner that BB′ shall be a straight line; and as, from the identity of the solids, BAE will then be equal to B′AE, it must follow, as we sought to demonstrate, that each of these angles is a right angle.

Fig. 120.



That $A_{,}'$ is also right angled may be proved as follows: From art. 50 we have

$$a'^2 - b^2 = -a^2$$
$$b'^2 - a^2 = a''^2$$
$$b^2 - b'^2 = c'^2$$

And adding these equations, there results

$$a'^2 = a''^2 + c'^2$$

Whence, art. 50, AEC is a right angled triangle, having the right angle at E.

* Referring to the scheme of notation, page 109, it will be seen that AB$=a$, AC$=a'$, AD$=a''$, BC$=b$, BD$=b'$, CD$=c$; the plane ABC$=$A, ABD$=A_{,,}$, ACD$=$A′, BCD$=$B, BCE$=$B′.

This characteristic property established, the remainder of the analysis will be obvious.

Thus, since

$$\frac{a'}{b} \cdot \frac{c'}{a'} = \frac{c'}{b}$$

we deduce

$$\cos. bc' = \cos. ba' \cos. ca' \ldots . (2)$$

And as, in art. 50–3, we find both the cosine of an angle to be less than unity, and to decrease whilst positive with the increase of the angle; and, further, that the negative cosines belong to angles greater than $\frac{\pi}{2}$: the equation 2, with these additions, will demonstrate both that cos. bc' is less than cosine ba', and, provided (ba') is always less than a right angle, that (bc') is greater than (ba').

This demonstration includes every possible position which the point D can assume in the line AD; and as the solids on either side of A, fig. 118, are identical, it follows that whatever is shown of ABCD will also apply to ABCD′; and, adding this remark to the preceding demonstration, we conclude, as a property of the type here chosen, that (ba') is less than the inclination of b to any other line in the plane CDD′, a property that will be found useful in defining the inclination of a line to a plane.

The inclination of the planes B and A′, or BCD and ACD, is justly measured, art. 38, by $(b'a'')$; whence we have,

$$\frac{a}{b} = \sin. ba'$$

$$\frac{a}{b''} = \sin. BA'$$

$$\frac{b''}{b} = \sin. bc$$

Dividing the first of which equations by the third, and comparing the quotient with the second equation, there results,

$$\sin.\ BA' = \frac{\sin.\ ba'}{\sin.\ bc}. \quad . \ . \ . \ (3)$$

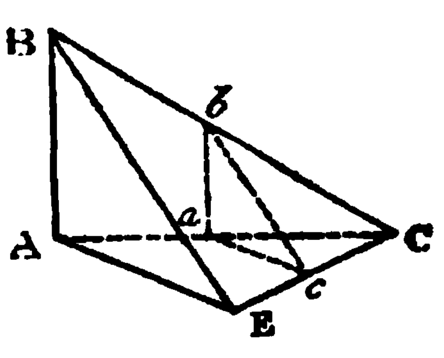
The construction employed sufficiently demonstrates that a rectangular pyramid could be formed by letting fall a perpendicular from D upon CB, fig. 119, whence, by analogy,

$$\sin.\ AB = \frac{\sin.\ b'a}{\sin.\ b'b}.$$

And from these equations combined with 2, and with the equations belonging to each of the faces, arts. 50, 53–1, &c. it will be easy for us to determine all that we at present require concerning the type.

But we must not omit, whilst employed on the properties of this solid, to show that it admits the idea of symmetry, which we have promised, art. 48, to prove of all figures.

Fig. 121.



The proof for the figure in question is derived immediately from what has been shown respecting right angled triangles: for taking Ca, Ce and Cb proportional to CA, CE and CB, and joining the points abc, we shall have

$$\frac{Ca}{Cb}=\frac{CA}{CB}=\cos.\ bCa,\ \frac{Ce}{Cb}=\frac{CE}{CB}=\cos.\ bCe,\ \frac{Ce}{Ca}=\frac{CE}{CA}=\cos.aCe;$$

whence it appears that bCa, bCe, and aCe are right angled triangles symmetrical with BCA, BCE and ACE,

art. 49 and 50. But from the properties of such figures, art. 50, we derive the equations

$$\frac{eb}{Cb}=\frac{\text{EB}}{\text{CB}},\quad \frac{ea}{Ca}=\frac{\text{EA}}{\text{CA}},\quad \frac{ba}{Cb}=\frac{\text{BA}}{\text{CB}};$$

and from these, again, we deduce

$$\frac{ba}{be}=\frac{\text{BA}}{\text{BE}},\quad \frac{ea}{be}=\frac{\text{EA}}{\text{BE}},\quad \frac{ba}{ae}=\frac{\text{BA}}{\text{AE}};$$

equations whence we conclude, art. 50, that *bea* and BEA are right angled triangles, the ratios of whose corresponding sides, and consequently whose corresponding angles are equal.

Such figures are symmetrical, and the ratio of *b*C and BC not being assigned, we see that any magnitude may be given to their sides, and consequently to the solids.

The equations 2 and 3, too, depending altogether on the angles at C, and the third of these angles being found when two of them are given, we conclude that two of the angles at C are data sufficient to determine the pyramid, and to inform us of all its properties.

62. But it will be recollected that we were led to the investigation of these properties not only as belonging to the standard solid, and merely on this account they would have occupied a place here; but also with the view of measuring, in the most appropriate manner, the inclination of a line to a plane. This inclination, we have already remarked, art. 57, might be estimated by the angle (bc'), which is the intersection of the plane A′ with a plane passing through b; but as (bc') is capable of innumerable values, we deferred making any convention with respect to the inclination in question, until we

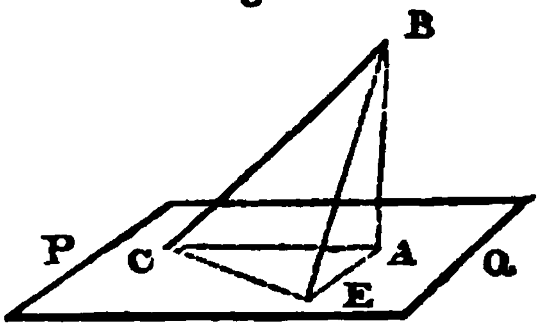
had examined the values which (bc') assumes, and determined whether any of them presented advantages peculiar to itself.

Where many quantities are offered in this way as measures of a simple quantity of a different kind, the greatest or the least among them are obviously to be preferred; neither can we in general hesitate in selecting the latter.

But we have shown both that any value of (bc') might represent the inclination in question, and also that (ba') is the least value which (bc') admits; we cannot doubt then as to the propriety of assuming (ba') to represent the inclination of b to A′.

According to this view of the subject, if it was required to measure the inclination of CB to the plane PQ, we should first suppose that out of an infinite collection of rectangular pyramids of every dimension, and possessing all the variety which such figures are susceptible of, a pyramid could be found such that, when placed on PQ, one of its edges could be made to coincide with CB; and, this done, we should then measure the inclination of the line to the plane by an angle BCA of the pyramid.

Fig. 122.



But from the first part of the article it is evident that an infinity of rectangular pyramids are capable of the adjustment here supposed, it might admit a doubt whether each of these possessed the same minimum angle BCA; but this doubt will be removed by considering that since all solid right angles, art. 46, are identical, we might construct all the pyramids in question out of the same solid right angle; and the equation 3 thus embrac-

ing the whole of these solids, its minimum angle would be their common base, and consequently the same in each.

But to give a wider range to the preceding demonstrations, we shall remark that c'' or CE, may be any straight line that passes through C and lies in PQ: for since $c'^2=a'^2+a'''^2$, we may, by increasing a''', cause c' to take every value from a' to infinity; and consequently cos. $c'a'=\frac{a'}{c'}$, to take every value between unity and zero; but in the development of the equations (1) already appealed to, we shall find that such values would correspond with the increase of $(c'a')$ from zero to $\frac{1}{4}$. And since, when $(c'a')$ was greater than $\frac{1}{4}$, its supplements would be less, these cases would correspond to those where E fell on the other side of A. And thus, whilst we consider CE as a side of a rectangular pyramid, we are still permitted to make it coincide with any straight line submitted to the restrictions above mentioned.

This remark will not only assist in giving their due extension to certain cases deduced as corollaries from the general proposition, but may also be used in stating that proposition under another form.

For since, art. 61, the plane B, if perpendicular to A' must coincide with the plane A, we conclude, by the assistance of the remark in question, that A is the only plane which fulfils, at once, the two conditions of passing through BC and being at right angles to PQ. And hence the propriety of the following definition.

To measure the inclination of a straight line to a

plane, let a second plane, perpendicular to the first, be passed through the given line.

The angle formed by this line, and the common intersection of the planes, will be the measure required.

From this agreement respecting the inclination of a line to a plane, we readily deduce the following corollaries, premising that we shall want their assistance in decomposing compound solids into rectangular pyramids.

63. 1. *If a line is at right angles to a plane, it makes right angles with every line meeting it in that plane.*

For it has been shown that c', (fig. 122) may represent any line lying in the plane A' and meeting b; but making (ba') (*equa.* 2) which measures (b A'), equal to $\frac{1}{4}$, we have cos. $bc' = o$, and $(bc') = \frac{1}{4}$. This corollary may also be proved from the fact that, in this case, the least angle formed by the line and plane is a right angle.

63. 2. *A straight line at right angles to two others which lie in a plane, is at right angles to the plane itself.*

For by varying the position of the line c' we may make it agree with either of the given lines; but unless (ba') (*equa.* 2) (which measures the inclination of b to the plane) is a right angle, only two positions of c' can make (bc') a right angle; and as these two positions occur when cos. (ca') is zero, they would belong, not to the two lines we suppose given, but to one and the same straight line. It follows, that when c' coincides with the lines in question, $(ba') = \frac{1}{4}$.

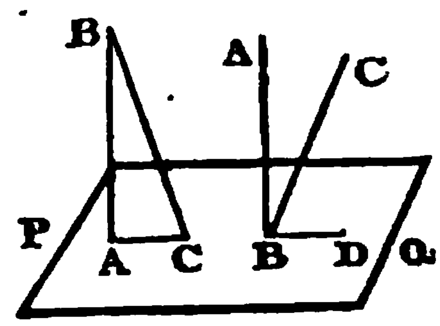
63. 3. *A plane which passes through a perpendicular to another plane, will be at right angles to this last.*

For in equation 3, making (ba') and bc each equal to $\frac{1}{4}$, we have $(\text{BA}') = \frac{1}{4}$.

63. 4. *Through the same point only one perpendicular to a given plane can be drawn.*

The proposition admits of two cases, according as B falls without the plane or within it; in either case let BA, BC, be two straight lines drawn through B perpendicular to the plane PQ. Let a plane be drawn through these lines, and imagine its intersection with PQ to be AC in the first case, and BD in the second.

Fig. 123.

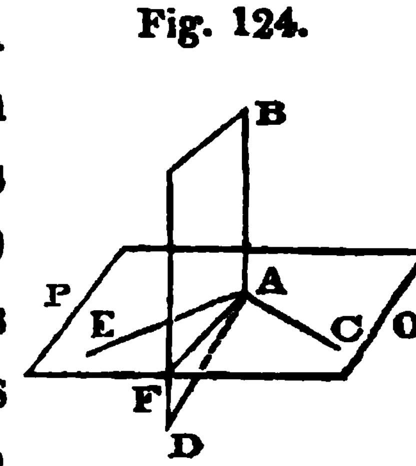


According to the former of these constructions, since BAC and BCA are both right angles, the three angles of the triangle BAC are greater than $\frac{1}{2}$; and according to the latter construction, since ABD and CBD are both right angles, the greater angle is equal to the less. Either result involving an absurdity, we conclude that AB and AC cannot both be perpendicular to PQ.

63. 5. *Straight lines perpendicular to the same straight line are in the same plane.*

Let AC, AD and AE be each perpendicular to AB; and causing a plane PQ to pass through AC and AE, let us suppose that AD does not lie in PQ. Through AB and AD describe the plane BD, and suppose AF the common intersection of BD and PQ. Then since BAF is a right angle by art. 63–2, and to BAD by hypothesis, it follows that BAF is equal to BAD, or the less angle is equal to the greater; and as such a

Fig. 124.



result is absurd, we conclude that AD cannot have a position different from AF, but must coincide with it and lie in the plane PQ.

The detailed examination of these particular cases of the inclination of a line to a plane, was introduced, as the reader has already been apprized, with the view of obviating doubts that might have arisen respecting the possibility of reducing compound solids, and comparing them with the type; but we must postpone the method whereby the comparison is effected, until we have enlarged our knowledge of angles by removing the most important out of the two deficiencies it yet presents.

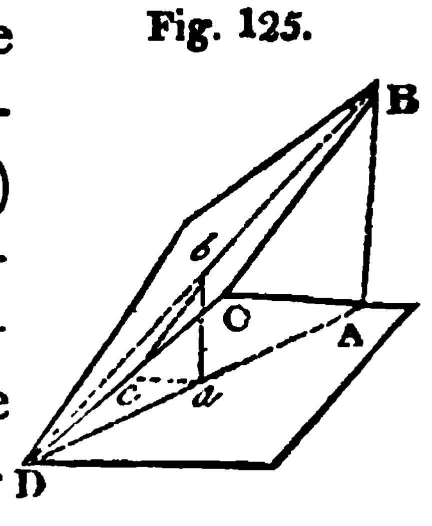
The angles formed by lines that do not meet, has not even been defined, and will occupy us in the next section; but we may add in *this* all that is wanting to confirm our notions respecting the inclination of planes.

64. The method of estimating such angles was mentioned in art. 38, and shown to be subjected to conditions, which we can now examine whether the measure there chosen fulfils.

Referring to the article in question, we shall observe that any choice which makes the angles m OA and n OA invariable, fulfils the first condition, and as these angles, in the measure we have selected, are right angles, the first condition will be satisfied.

To examine the second let us apply between the planes a rectangular pyramid BADC; the base of this pyramid may be made to coincide with the lines Om, On, (fig. 92) and consequently with the plane passing through those lines: the intersection OA will then necessarily agree with CD, for otherwise two perpen-

Fig. 125.



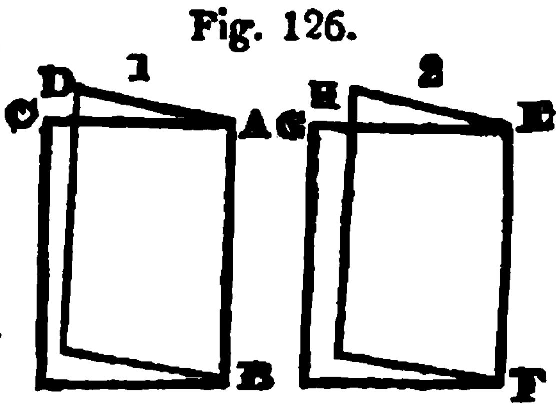
diculars could be drawn from the same point to the same plane, which we know to be impossible: but the line CD falling on OA, and CB on OM (fig. 92), the face BCD of the pyramid must coincide with the plane ABCD (art. 38): and in the same way it may be shown that ADEF, fig. 92, coincides with the face DCA.

The application we have supposed is therefore possible, and by art. 61 we might apply in the same way a second rectangular pyramid D*abc*.

But either of the angles BCA or *bca*, which have been shown, art. 61, to be equal, will measure the inclinations of the planes; and as the point *e* was taken arbitrarily in the line CD, we conclude that wherever the point C of fig. 125, or O of fig. 92, may be chosen, the measure of the inclination will remain the same.

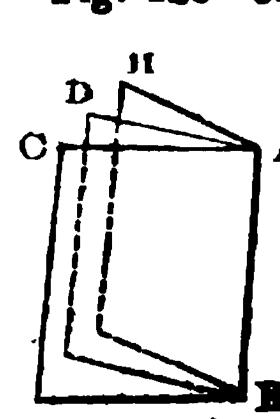
The third condition may also be examined by means of the measure we have chosen for the inclination of a line and plane.

Fig. 126.



For the inclination of any two planes A and A, (or ABC and ABD), is identical with the solid angle, (art. 44 and 46) they inclose; and we have only to examine whether the more convenient measure assumed in art. 38 varies in the same ratio with the variations of this solid. But taking a solid angle GEHF, identical in all its parts with the former, and causing the plane E of this second prism to coincide with the plane A, of the first, we shall, in this way, construct a new solid angle that is evidently double to either of the angles from whose addition it resulted. If then the measure

Fig. 126—3.



of art. 38 is correct, the inclination of the planes in the compound solid should also be double of the corresponding inclination in either of the two solids from which it is compounded.

But drawing from A and E lines AC, AD, EG and EH perpendicular respectively to AB and EF, and lying in the planes whose inclinations are to be measured; the equal angles (art. 38) CAD and GEH will be the measures of those inclinations: and since, fig. 126—3, the lines AC, AD and AH are perpendicular to AB, they lie in one plane, and the equal angles CAD, DAH, add together and form an angle CAH, which is double of CAD.

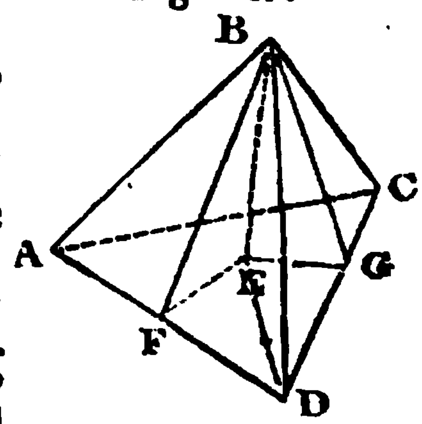
But CAH measures the inclination of the planes A and $A_{,}$ (or ABC and ABH), and CAD measures the inclination of A and $A_{,}$; and thus as far as this particular case is concerned, the measure we have chosen is found to answer the third condition. But the reasoning, though applied to a particular example, is evidently general, since, however many such prisms we add, the angles which measure the inclination of their planes will be added together at the same time, and the compound solid and the compound measure will increase in the same proportion.

It might justly, indeed, be objected to this reasoning, that all solid angles cannot be formed by successive additions of the same unit; but it has been proved before, in the elements of algebra, that when quantities are measured by others of a different kind, the objection here alluded to is of no importance; and that a demonstration which extends to quantities formed by the successive additions of an indefinite unit, is in fact perfectly general.*

* Note 2.

65. With this confirmation of a measure which has entered so largely into our idea of the rectangular pyramid, we shall find no difficulty in applying that solid to the purpose for which it has been investigated, and need but a few words to show how readily all compound solids can be reduced to this simple type. Taking, for example, any triangular pyramid ABCD, we may apply to one of the solid angles D, a rectangular pyramid BEFD; and since the angles BDF and BDE are given, it is evident that only one such pyramid can be found. But applying to this a second rectangular pyramid BDEG, in such a manner, that whilst the base rests on the face A′ the sides DE shall be common, it follows, that not only the sides EB coincide, but the faces passing through DE and EB will coincide also.

Fig. 127.



So far either of the rectangular pyramids is subject to limitation, but the condition which restricts the second solid, merely assigns the side DE, and is not sufficient to determine the angles; and hence, among such solids a rectangular pyramid BDEG may be found, that, placed in juxtaposition with BDEF, shall complete the solid angle at D.

The equation 2 applied to those solids will determine the angles FDE, GDE and BDE, of which the last measures the inclination of b' to the plane A′; and when these are known, the equation 3 will determine the inclination of the planes.

The construction we have applied to the angle D may also be used for either of the other angles; and thus we

are enabled to compare every part of the solid with the triangular pyramid, and to obtain equations whence all the relations of the figure may be found.

66. Whilst the three plane angles at D remain the same, the solid angle they include cannot vary: for as these angles are data sufficient to determine, art. 61, the inclinations of the planes in which they lie, it is obvious that if another solid angle was included by three plane angles equal to those at D, and similarly disposed, the two solid angles, when superposed, would coincide, and be equal in all respects.

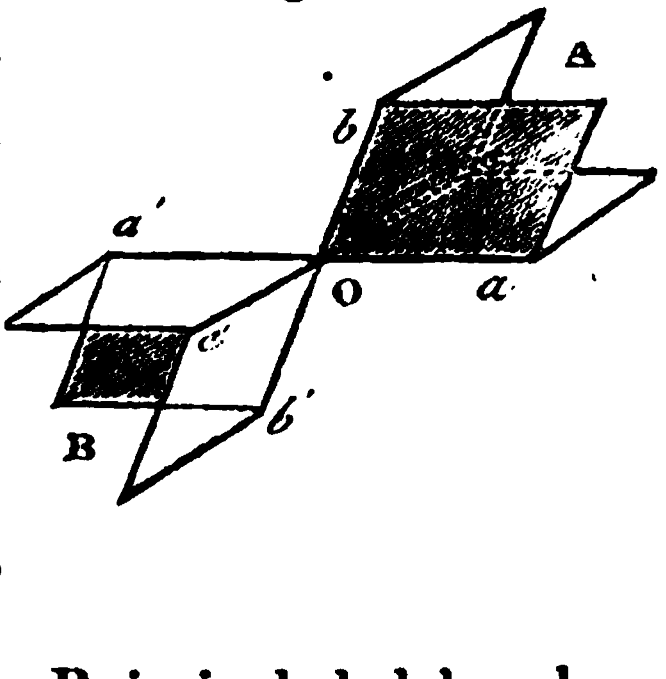
A knowledge of this fact will enable us to supply the demonstrations promised in art. 46.

In that article we had proposed to investigate the relation between the solid angles included by two, and those included between three or more planes; and, as a preliminary step, to show that opposite, or vertical, solid angles were equal.

The latter proposition is merely a corollary from that we have just demonstrated.

For suppose A and B two such angles, that is, suppose the solid angle B formed by producing the planes that include A. It is evident that for each of the plane angles that include A, there will be an equal and opposite plane angle among those which include B: thus the angle $a'Ob'$ is equal and opposite to aOb, and $c'Ob'$ to cOb, &c.

Fig. 128.

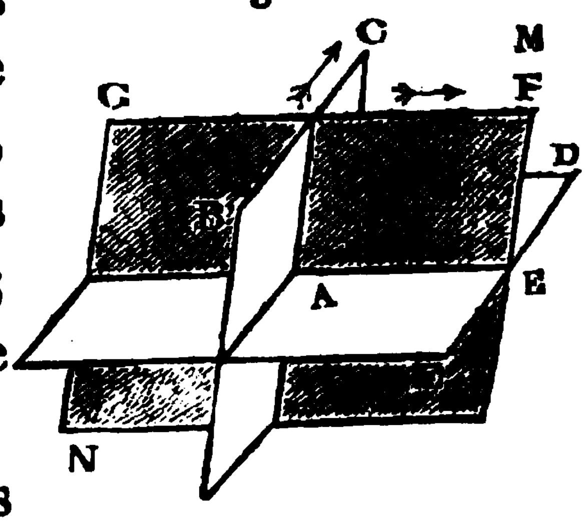


Since, therefore, the angle B is included by plane

angles equal to those which include A, and similarly disposed, by what has been shown above, the solid angles B and A are equal.

67. The relation between the solid angles included by "two," and those included by "three" planes, is an immediate consequence of this equality of opposite angles. The planes A, A″, A‴,* for example, when produced in all directions will include eight solid angles; of which four lie above the plane A″, and four below.

Fig. 129.



The notation of page 109 does not enable us to distinguish these angles apart, but regards any one of the eight as the angle (AA″A‴); the objection that would arise to this indistinctness in the notation will hereafter be removed, but it will be sufficient for our present purpose if we regard the angle (AA″A‴) as representing M, a solid angle which agrees with that denoted by A in the preceding article.

A similar remark applies to the angles included by any two of the planes; the angle (AA″), for example, will mean either of four angles; but here also the difficulty can be removed by pointing out the side towards which the angle we are considering lies.

Denoting then by (AA″), (A″A‴) and (AA‴) those solid angles which each contain M:

It is evident that, substituting for the third (AA‴), its equal and opposite solid angle, the sum of the three will not only complete the whole space above the plane

* Or ABC, ADE, AFG.

A″, but also the spaces occupied by N and by M; which last is twice included in this sum.

Hence, we have the equation,

$$(AA'') + (A''A''') + (AA'''') = \tfrac{1}{2} + M + N.$$

But M and N, being vertical angles, are equal; and, by what was observed above, M is the angle $(AA''A'''')$.

With these substitutions the equation becomes,

$$(AA'') + (A''A''') + (AA'''') = \tfrac{1}{2} + 2(AA''A''').$$

Whence,

$$(AA''A''') = \tfrac{1}{2}(AA'') + (A''A''') + (AA'''') - \tfrac{1}{4},$$

or, a solid angle contained by three planes is equal to half the sum of the inclinations of those planes minus a right angle.

The solid angle included between two planes has been measured, art. 38 and 64, by a plane angle; but as the two measures agree, the proportion we have demonstrated will remain the same which ever method of measurement is followed.

PRELIMINARY REFLECTIONS TO SECTION V.

The type of closed figures, and the type of closed solids, have each had their properties reduced to algebraic equations; but to render these results extensively useful, the process of compounding all figures from their types should also be represented by equations.

INQUIRIES SUGGESTED BY THESE REFLECTIONS.

Closed figures and solids being compounded of their respective type, it is required to discover so uniform a method of effecting this composition, that in all cases, the results may be represented by the same equations.

SECTION V.

GENERAL METHOD OF COMPARING ALL FIGURES WITH THE TYPE.

Use of parallel lines and planes as instruments to compare compound figures with the right angled triangle, and compound solids with the rectangular pyramid—theory of parallel straight lines—theory of parallel planes—inclinations of lines that do not meet—general method of comparing linear figures with their type—elementary proposition on which this analysis is founded—property of closed figures which expresses the comparison sought.

68. Examples of the decomposition of figures have been given in articles 51 and 65, where triangles and triangular pyramids were respectively compared with the type of linear and of solid figures; but although the instrument employed there, the principle of superposition, is of extensive use, our analysis would not be considered as having the characters of a ready and efficient method, unless its application were uniform as well as general.

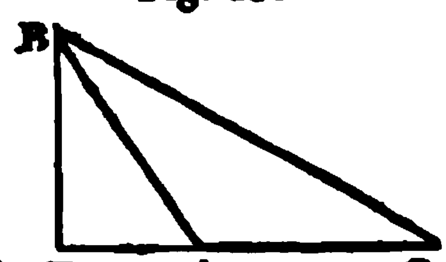
To effect this object let us briefly review the method of decomposition we have employed.

Chap. I. First principles of the science obtained from the relations of a finite number of points.

Art. 68. Use of parallel lines and planes as instruments to compare compound figures with the right angled triangle, and compound solids with the rectangular pyramid.

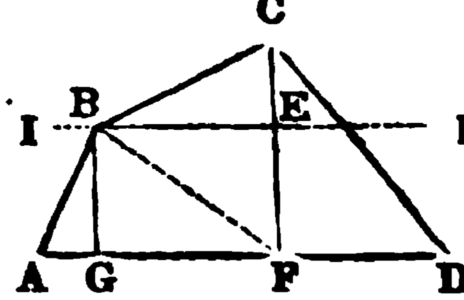
Referring to art. 51, we find the decomposition of the figure examined there, effected by right angled triangles which are arranged upon a common base. This base, fig. 113 and 114, passes through the point A, and is indeed a continuation of the side AC: but such an arrangement no longer suffices when we attempt to apply the same method of analysis to a more complicated figure.

Fig. 130.



In analysing, for example, the polygon ABCD, we may cause base lines to pass both through A and B, and arranging right angled triangles on these lines may perhaps succeed in filling up all the space included by the figure. But the process will evidently present in this example many difficulties not met with in the former. For placing ABG and CDF upon the base AD, and BCE upon the base IK, we are uncertain, in the first place, whether CBE and CDF will coincide in CE; and, secondly, admitting this coincidence, we have doubts concerning the nature of the figures BGF and BFE.

Fig. 131.



To remove these doubts we must attend to the properties of the bases whereon our triangles are arranged, and, to obtain that uniformity of operation which we seek, must cause these lines, AD and IK, to be as nearly identical in all their relations as is consistent with the condition of their passing through different points, through A and B.

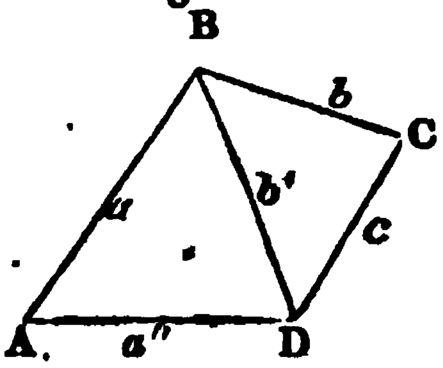
Straight lines possessed of this relation to each other are termed *parallel straight lines;* and the investiga-

tion of their theory is obviously the next step in our analysis.

69. 1. The theory of parallel straight lines is in fact implicitly included in the equations (1) already obtained for triangles; but as this method of treating the subject would require us to introduce the idea of infinity, we shall prefer obtaining the properties of these lines from the relations of four points, which indeed seems a more natural source for them.

Suppose, then, the points ABCD arranged in one plane, and with such relative positions that BAD+CDA, or $aa''+ca''$, shall be equal to $\frac{1}{2}$.

Fig. 132.



From this condition united with what has already been shown respecting triangles, we have

$$\begin{aligned} aa'' + ab' + b'a'' &= \tfrac{1}{2} \\ bc + bb' + cb' &= \tfrac{1}{2} \quad \ldots\ldots \quad (4) \\ aa'' \qquad + ca'' &= \tfrac{1}{2} \end{aligned}$$

Subtracting the last of these equations from the sum of the two first, there will arise

$$bc + ab = \tfrac{1}{2} \quad \ldots\ldots \quad (5)$$

or the sum of the two interior angles which the lines a and c make with b will be equal to two right angles.

And putting the second of the equations, 4, under the form

$$bc + ab - ab' + cb' = \tfrac{1}{2}$$

and substituting for the two first terms their value obtained from equation 5, there results

$$cb' = ab' \quad \ldots\ldots \quad (6)$$

and if B and C are so assumed that

$$ab + aa'' = \tfrac{1}{2}$$

we shall have, by subtracting this equation from 5,

$$bc = aa'' \; . \; . \; . \; . \; . \; 7$$

69. 2. From the equation 5, we learn that when a right line P, fig. 135, meeting two other right lines M and N, makes the two interior angles together equal to

Fig. 133.

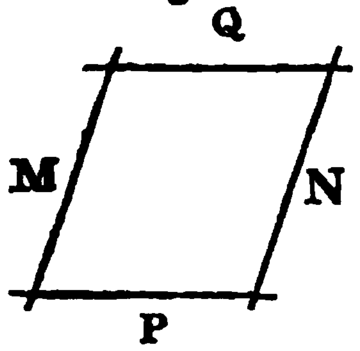


Fig. 134.

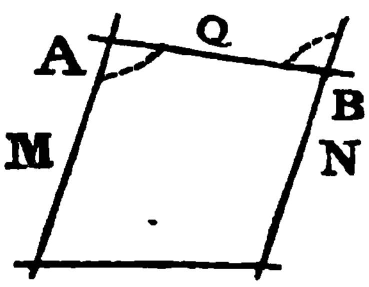
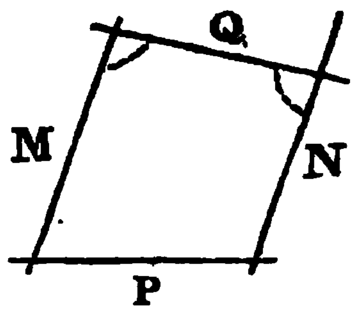


Fig. 135.



two right angles, any other line Q, meeting the same lines, will also make the sum of the two interior angles equal to two right angles.

69. 3. From the expression 6 we learn the equality of the alternate interior angles MAB, ABN′, fig. 134.

69. 4. And as the equations 6 and 7 prove the angles of the triangle ABD, fig. 132, respectively equal to those of the triangle BCD; and as the side BD is common, BC is equal to AD.

Combining this result with the remarks above made, and assuming M and N as before, it appears that if P and Q, fig. 133, are inclined to either of these lines at the same angle; they will also be equally inclined to the remaining line, and the parts of P and Q intercepted between the lines M and N will be equal.

Such a figure is called a *parallelogram*, and from

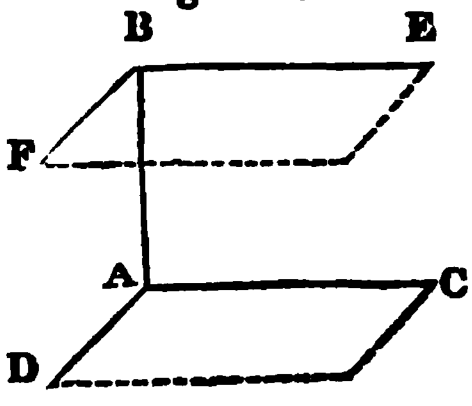
what has been said, it appears that if any three of its angles are right angles, the fourth angle will also be a right angle; and if three of its sides are taken equal, the fourth side will be equal to either of the others. A quadrilateral of this kind, namely, which has all its sides equal, and all its angles right angles, is called a *square*. Part I. art. 13.

69. 5. As these lines M and N appear, from the last remark, to be every where at the same distance, they must be so disposed with respect to each other, that, produced indefinitely, they cannot meet: and, accordingly, parallel lines might be defined, if a definition were necessary, either as straight lines lying in one plane, and equally inclined to a given straight line; or, as straight lines that lie in one plane but do not meet, however far produced.

70. 1. And transferring these definitions from lines to planes, we may consider parallel planes either as planes that, produced indefinitely, do not meet; or, as planes equally inclined to a given straight line.

Adopting the former definition, the theory of parallel planes might be derived from the equation (3), but, perhaps, the simplest method of showing their existence and properties is to conceive a plane passing through two lines, BE and BF, that are at right angles to AB, a perpendicular to the plane CAD.

Fig. 136.



With this construction, we observe, that since BA is perpendicular at the same time to each of the planes A′ and B″, it is perpendicular to all straight lines which meet it in those planes.

Now if these planes were not parallel they could have a point in common: but this is obviously impossible, since joining such a point with A and B, we could then form a triangle having two right angles.

70. 2. And nearly in the same way we may prove the converse proposition, that every plane parallel to A′ will be at right angles to AB, a perpendicular to A′.

For drawing through AB the planes ABE and ABF, let their intersections with A′, and its parallel, be BE, BF, AC and AD. Then, since BE and AC are in the plane ABEC, either they are parallel, or, produced sufficiently far, they will meet; but if BE and AC meet, their point of intersection will be common to the planes FBE, DAC, which is absurd, since these planes are parallel: whence we conclude that BE and AC are parallel lines.

But BAC is a right angle, and since BC and BE are parallel, ABE is also a right angle. In the same way it may be shown that ABF is a right angle. Hence AB is perpendicular to two straight lines BE and BF, and consequently to the plane passing through them.

And from this proposition we may deduce the following corollaries.

1. Through the same point only one plane can be drawn parallel to a given plane.

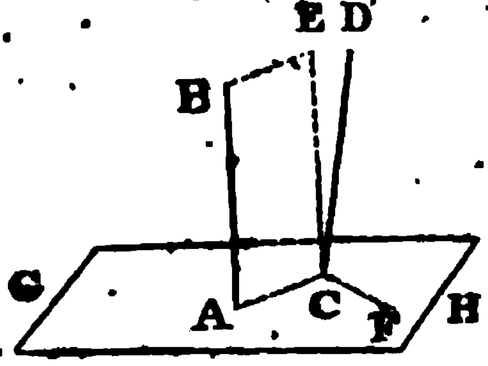
2. Planes parallel to the same plane are parallel to each other.

3. The intersections of parallel planes with a third plane are parallel.

For the intersections, lying in planes that do not meet, cannot themselves meet; whence, as the intersections also lie in one plane, they must be parallel.

4. Perpendiculars to the same plane are parallel, and of parallel lines, if one is perpendicular to a plane, the others are also perpendicular to it.

Fig. 137.



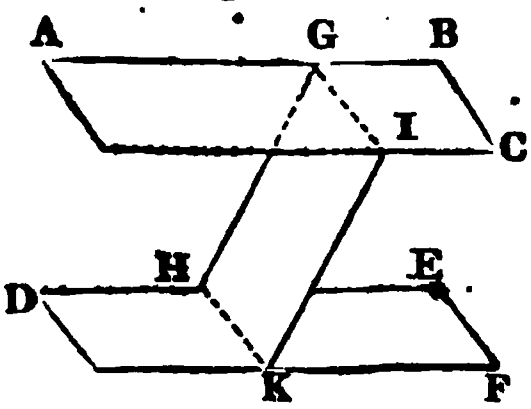
Let AB and CD be two of the perpendiculars. Through AB and the point C draw a plane ABCE; it will be perpendicular, art. 63–3, to the given plane GH. In these two planes, respectively, draw CE and CF, each, at right angles to AC. By art. 38, ECF is a right angle; and EC, making right angles with the two lines AC, CF, is at right angles to the plane GH, wherein they lie. It follows that CD and CE must be identical, or otherwise there would be two perpendiculars to the plane from the same point in it; but CE, with which CD is identified, lying in the same plane with AB, and making the angles at A and C right angles, is parallel to AB.

To prove the second part of the proposition: let AB and CE be two of the parallel lines, of which AB is perpendicular to GH: and completing the construction as before.

BAC + ACE is equal to $\frac{1}{2}$, and BAC is equal to $\frac{1}{4}$, wherefore ACE = $\frac{1}{4}$. But ECF is also equal to $\frac{1}{4}$. Whence CE, being perpendicular to two lines which lie in the plane GH, is perpendicular to the plane itself.

5. This third corollary may be used to show that parallel planes intersected by parallel lines, cut off equal portions of them.

Fig. 138.



Assuming the planes to be A and D, and the parallels G and I, we can suppose through the latter a plane G to be passed, whose intersections with A and D will be parallel; and the lines G, I, G′, H′, thus forming the sides of a paral-

lelogram, any two which are opposite will be equal, and the line G will be equal to the line I.

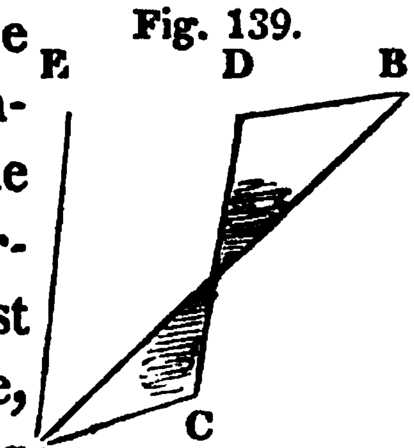
71. The definition of parallel lines that we have given in art. 68, and which regards them as lines having all the agreement of direction that is consistent with their passing through different points, leads immediately to the solution of a problem hitherto deferred. I allude to the measure of lines that cross each other without meeting.

Such lines, however they are turned, can never have the same direction, and the nearest approach which they can make to such an agreement of direction, is by becoming parallel.

In estimating therefore the inclination of lines of this kind, we are not to calculate their departure from a perfect agreement, but their departure from the nearest agreement of which they are susceptible; that is from those directions wherein they would be parallel.

Drawing, then, from a point, A, in one of the lines, AB, a straight line, AE, parallel to the other, CD; the plane angle, EAB, included between the first line and the parallel to the second, will be the departure of the two lines from the nearest agreement of which they are susceptible, and will therefore justly represent their inclination.

Fig. 139.



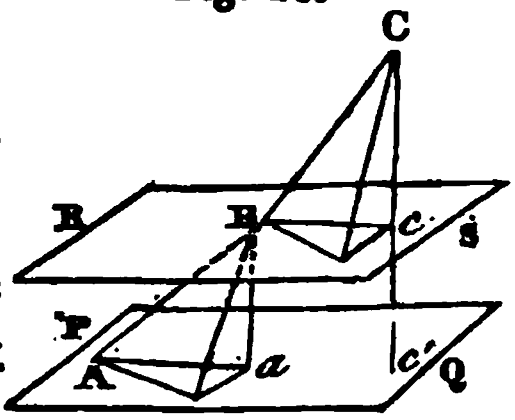
72. With the assistance of the properties of parallel lines and planes which we have thus obtained, it will not be difficult to devise a uniform method of comparing figures with their type.

For, in the first place, we have seen, art. 62, fig. 122, that drawing a plane through any point C of a given line CB, we can place on this plane a rectangular pyramid that shall have its hypothenuse coincident with CB.

The triangle BCA will then be right angled at A; and BA, the distance of B from the plane PQ, will be equal BC cos. CBA.

And, in the same way, if two consecutive lines, AB and BC, were given, we might cause planes to pass through *two* points, A and B, and, by the process we have described, obtain B*a*, the distance of B from PQ; and C*c*, the distance of C from RS. A result which is evidently a considerable step towards obtaining the ultimate object of our present research, namely, an equation containing the relations of BA and BC. For if the planes PQ and RS were drawn parallel to each other, we should have C*c*′ = C*c* + *cc*′= (art. 70—5) C*c* +B*a* = BC cos. BC*c* + AB cos. AB*a*: an equation which will be wholly expressed in terms of AB, BC and their inclinations, provided we can express C*c*′ in such terms.

Fig. 140.



Before we inquire concerning this last condition, let us extend what has already been said to the general case, wherein, instead of two given lines AB and BC, we have any number of given lines. Our proposition will still be a mere form of the elementary truth, that the whole is equal to the sum of its parts, and may be thus enunciated.

73. *If parallel planes are drawn through any number of points, and the latter are reckoned in sequence,*

the sums, or in some cases the differences, of the distances of each point from the plane immediately preceding it, will be equal to the distance of the last point from the plane first drawn.

Fig. 141.

Considering the points A, B, C, D, E, in the order they are written, fall from each a p on the plane imme ceding, that is, o which passes throu superior point, we by art. 70—5,

$$Ee' = Bb' + Cc' - Dd + Ee \quad . \; . \; . \; . \quad 8$$

a result that may evidently be extended to any number of points, and which therefore agrees with the enunciation.

This simple proposition was obtained by placing a rectangular pyramid on each of the parallel planes; for, by art. 62, it is in this way that we assure ourselves of the possibility of letting fall the perpendiculars above mentioned; but, when this construction is accomplished, the analysis proceeds entirely by right angled triangles; and will assume a more convenient form by supposing an arbitrary line m, drawn perpendicular to the several planes. We have then,

$Bb = AB.\cos.ABb$

$Cc = BC\cos.BCc$

$-Dd = -CD\cos.CDd = +CD\cos.\text{supp}.CDd$ (art. 51)

$Ee = DE\cos.DEe$

And observing that with respect to any of these angles,

CDd excepted, the said parallel to m is estimated in the same direction, we shall perceive a reason for choosing the supplement of CDd, instead of the angle itself.

Examining, for example, the angles ABb and BCc, which are formed by AB, Bb and BC, Cc, we observe that Bb and Cc, the sides parallel to m, are both estimated in the same direction; whilst, with respect to the angle CDd, formed by CD and Dd, this direction is reversed: and as a uniform method of proceeding is always to be observed in analysis, we shall be justified in assuming, instead of the angle formed by CD and Dd, that formed by the first of these lines and the prolongation of the second; or, which is the same thing, we shall be justified in introducing the supplement of CDd instead of the angle itself.

This artifice of calculation we have already used in art. 51, where it appeared that we may always substitute, for the cosine of an angle, the cosine of its supplement, provided we place before the latter a negative sign.

Adding, then, the equations we have just deduced, and putting for the sum of the left hand column its value Ee', there results

$$Ee' = AB \cos. ABb + BC \cos. BCc + CD \cos. \text{supp. } CDd + DE \cos. DEe$$

or

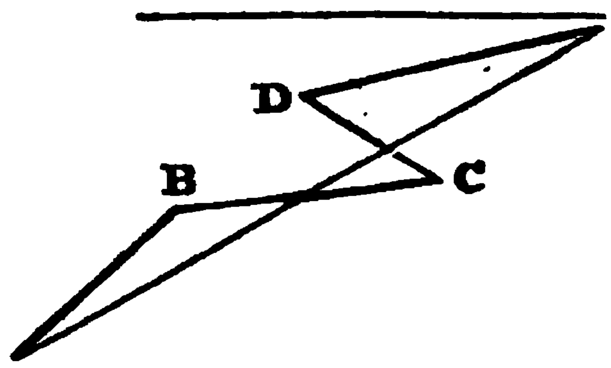
$$Ee' = a \cos. am + b \cos. bm + c \cos. cm + d \cos. dm$$

74. We may now examine the condition mentioned in art. 72, where a certain line Cc' was to be expressed in terms of the given lines and their inclinations: this line Cc' expressed the distance between the two extreme planes, and corresponds, therefore, with the line Ee' of the article we have last considered. But to express this

line in such terms as are required, we must have recourse to closed figures, to the theory of which, it has already been remarked (art. 47), all geometrical analysis can be reduced.

The lines AB, BC, CD, DE do not, of themselves, form a closed figure; but, drawing a line from A to E, such a figure is formed by the lines AB, BC, CD, DE and EA, fig. 142, and we immediately have (figs. 141 and 143) $Ee' = AE \cos. AEe' = a''' \cos. a'''m$. And equating this value of Ee' with that deduced in the preceding article, we obtain the equation

Fig. 142.



$$a''' \cos. a'''m = a \cos. am + b \cos. bm + c \cos. cm + d \cos. dm \quad . \; . \; . \; . \; . \; . \quad (9)$$

And as the method of proceeding will remain the same whatever is the form of the polygon, or the number of its sides, it is evident that we have accomplished the object of this section, and discovered a regular and uniform method of decomposing closed figures into right angled triangles.

PRELIMINARY REFLECTIONS TO SECTION VI.

Our idea of distance, or *linear quantity*, was obtained by considering the two given points that determined the direction of a line, as its boundaries. But lines themselves are the boundaries of planes, and planes the boundaries of solids.

INQUIRY SUGGESTED BY THESE REFLECTIONS.

By considering, then, lines as the boundaries of planes, should we not obtain an idea of *superficial quantity*; and by considering planes as the boundaries of solids, an idea of *solid quantity?*

These three species of quantity, being alike parts of space; ought they not to have some relation to each other?

Lastly, what modifications will these new ideas require in the analysis of the preceding sections.

SECTION VI.

OF THE DIFFERENT SPECIES OF QUANTITY WHICH ARE INCLUDED IN THE RELATIONS OF A FINITE NUMBER OF POINTS.

The three dimensions of space—of space which possesses but two of these dimensions—its unit—of space which possesses the three dimensions—its unit—five units employed in geometry—area of the type of plane figures—solidity of the type of solid figures—the general analysis of figures extended to their relations of area and solidity.

75. Our first acquaintance with the relations of space, we observed in the introduction, was obtained by a process contrary to that usually employed in teaching the same principles. From external objects we first learned the properties of definite portions of space, and thence, by an obvious transition, passed to surfaces, to lines and points. And this method, which experience shows to be the most natural, is, at the same time, the most scientific, for without an adequate idea of space, either as a whole, or by its sensible portions, what intelligible notion could we form, either of a plane, or of that principle of superposition which is the basis of geometric reasoning.

But, though such is undoubtedly the proper method of teaching the first elements of geometry, there is yet a stage of advancement, where, retracing our steps, and commencing with the abstract idea of space, and of points which are the symbols of position, we should proceed gradually to reason from points to lines, from lines to surfaces, and from surfaces to solids. Such is the method we have followed in the preceding pages, and which, leading us from simple directions to comparative, and thence to directions possessing the common property of meeting when produced, will at last teach us to regard space as extended according to three infinite, and, in some measure, arbitrary dimensions.

The right lines BB′, CC′, DD′, art. 58, which were shown to be mutually perpendicular, will enable us to form just ideas of these dimensions, and will connect their properties with those of the pyramid chosen as the type of solid figures, and whose relations were deduced in the article referred to.

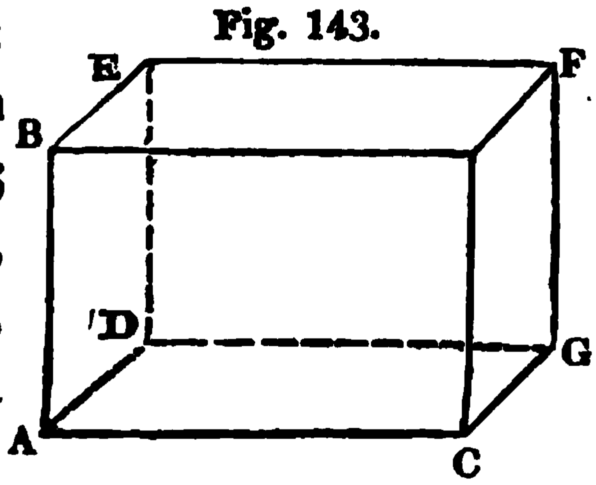
The general notions we are supposed to possess concerning space, will assure us that the point A and the direction BB′ may be assumed at pleasure; but this done, the direction CC′ is subjected to a condition; since, according to what we have learned concerning the angles formed by lines and planes, CC′ must lie in a plane passing through A and perpendicular to BB′, and only one such plane, CDC′D′, can exist.

When this condition is fulfilled, the direction of CC′ is, otherwise, arbitrary; but CC′ once assumed, the remaining direction DD′ is completely determined, since the definition we have given requires that it should be

drawn in a given plane, perpendicular to a known line lying in that plane, and through a given point.

But one of the clearest illustrations the three dimensions of space admit of, is obtained from those solids whose angles are right angles: the existence and nature of such figures may be readily shown; for, having proved, in art. 58, that three planes, A, A′ and $A_{,,}$ can be so drawn as to enclose a solid right angle, we have merely to conceive three other planes drawn at any distances from the former, but respectively parallel to them; and the six planes taken together will enclose a portion of space which has been termed the *rectangular parallelopiped,* and whose angles are solid right angles.

Fig. 143.



As the opposite edges of this solid are formed by the intersections of parallel planes with a third plane, art. 70–2, cor. 3, 5, they are, themselves, equal and parallel, and whatever is asserted of the edges having their vertex at one angle, will be also true of edges having their vertex at another; and thus three sides having a common vertex will be mutually perpendicular, and may be taken to represent the three dimensions of space.

The reader cannot fail to observe the resemblance between the rectangular parallelopiped and objects he is daily conversant with, and will perceive the three dimensions of space to agree with the ideas we denote by *length, breadth* and *thickness.*

The length of the solid is independent of the breadth, and the breadth of the thickness; we may therefore, if

we please, make all these dimensions equal; in which case the parallelopiped is called a *cube*.

76. When points are so arranged that only two of the dimensions of space are formed among their relations, the points lie in a plane passing through those dimensions; and the analysis of the relations in question is said to belong to *plane* geometry: whilst we refer to *solid* geometry the relations of points whose relative positions involve all three of the dimensions of space.

The route we have taken has not led us to make so marked a distinction between these two species of geometry as most writers have done, but rather has induced us to regard one species as a particular case of the other. This subject, however, will be more fully illustrated hereafter, and in the mean time it will be necessary to point out some simple elements that are suggested by these ideas respecting planes and solids.

Employing, according to what has been said in the preceding pages, the theory of closed figures as the instrument of investigation, we are naturally led to compare together the portions of space included in such figures, and thus arrive at the idea of an *area*, or a portion of a surface included in a closed figure. These portions, occupying hereafter a place in our analysis, must be estimated according to the law required in measuring quantity, namely, by a comparison with a unit of their own kind.

77. That such an unit is altogether independent of the standard assigned to linear measure, will be imme-

diately perceived when we consider that no possible addition of lines, which are mere directions, can give rise to a species of quantity possessing a dimension not found in lines. It will also be seen that the unit of areas, or superficial measure, is as arbitrary in its nature as the linear unit; and in selecting it we need be guided by no other consideration than a due regard to practical convenience.

The use we have made of the right angled triangle in analysing the relations of points would suggest for our present purpose the right angled triangle having both its sides equal to unity; but geometers, guided by obvious reasons, have preferred as the unit of superficial measure the double of this triangle, or *the square whose side is a linear unit.*

78. It will not be necessary, after what has been here said, to enter into much detail respecting quantity of three dimensions, or the definite portions of space cut off and bounded by surfaces; an argument analogous to that above used will assure us that quantity of this kind cannot be formed by an aggregation of lines or surfaces; whilst reasons similar to those alluded to as obvious, have caused the *cube*, a rectangular parallelopiped whose sides are equal, to be chosen as the standard reference; the side of this cube, as in the preceding case, being the unit of linear measure.

79. Returning upon our steps and reviewing the subject which occupied us in this section, we shall add some additional remarks illustrating more fully the connection and common origin of the five units of quantity.

Chap. I. First principles of the science obtained from the relations of a finite number of points.

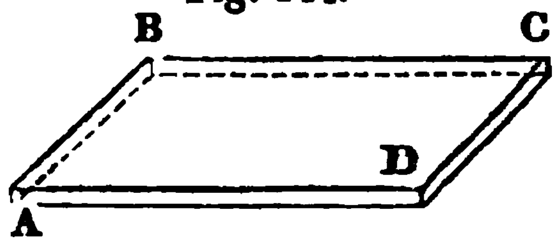
Art. 79. Five units employed in geometry.

These it will be recollected were the standards of comparison assumed for the measurement of solids, solid angles, plane areas, plane angles, and lines. Now, different as these quantities seem to be, they are all parts of that infinite whole which we understand by the term space, and thus, as parts of the same existence, are not altogether so heterogeneous as some authors seem to regard them.

The first of these quantities, solids, are real parts of the infinite whole alluded to, but parts too small to bear any comparison with infinity: whilst solid angles, which may be measured by the solid they include, are definite and assignable portions of space, considered as infinite in all its three dimensions.

Plane areas, the third species of quantity enumerated, bear the same relation to an infinitely extended plane, which solids bear to space in general, and a similar relationship subsists between plane and solid angles: whilst lines, our last species of quantity, are to an infinitely extended line what areas are to plane space, and solids to space of three dimensions.

Fig. 144.



And whether we regard an infinitely extended plane as a limit separating the parts of space on one side from those on the other, or as a solid indefinite in two dimensions and but little extended with respect to the third, we shall want but this further inquiry to arrive at the same conclusions respecting its relations, namely, to determine with certainty the effect which the third dimension has on the relations of the solid, and to trace what these relations would become when the third element was altogether omitted.

Sect. VI. Of the different species of quantity which are included in the relations of a finite number of points.

Art. 79. Five units employed in geometry.

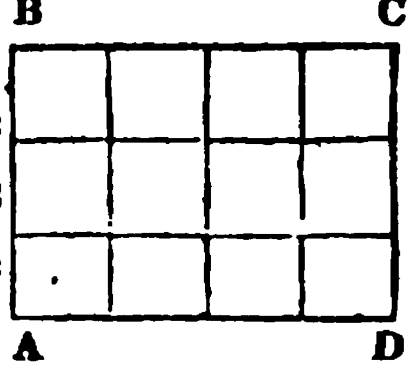
To illustrate this subject further, let us suppose the plane spoken of to be a smooth sheet of paper; the diagrams and lines drawn there are not perfectly in one plane, since, examined by a microscope, the paper appears full of asperities, and even the pressure of the hand, operating on so soft a substance, causes the lines to sink deeper in some places than others; nevertheless we reason respecting such diagrams by the rules of plane geometry; and in practice regard our results as sufficiently applicable to the actual diagrams, although obtained by a mental abstraction of their irregularities. But substituting for the paper a plate of polished metal, the errors we speak of become less; and as there is no limit to the excellence of temper and polish which we have a right to assume in our materials, it is obvious that the relations of space of three dimensions pass by insensible gradations into the relations of space having but two, or only one, dimension.

Assuming, then, the whole of space, or space infinite in three dimensions, as the standard of comparison, the five species of quantity which we have enumerated would be referred to a common unit. The symbol of solid angles would be a proper fraction; that of plane angles a fraction with an infinite denominator; whilst the symbol of definite lines would involve an infinite divisor of the fifth degree. But, avoiding the further prosecution of this subject, as unimportant, I shall devote the remainder of this section to the fundamental theorem developed in art. 74; applying to it the new ideas we have obtained; and seeking, by their assistance, to extend its application, until this includes, not only the linear relations of a

definite number of points; but their relations of solidity and superficies.

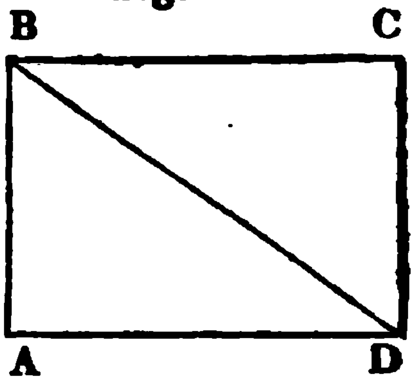
80. 1. By dividing two adjacent sides of a rectangular parallelogram into linear units, and drawing through the points of division lines parallel to the remaining sides, we shall, by art. 69, sub. div. 1, 2, 3 and 4, decompose the figure into squares each of whose sides is a linear unit; and hence, art. 77, the number of superficial units in the rectangular parallelogram, will be equal to the number of these squares; or to the product obtained by multiplying the number expressing the linear units in any one side, by the number expressing the linear units in either of the sides adjacent to it.

Fig. 145.



80. 2. A diagonal drawn in a rectangular parallelogram, that is, a straight line drawn from one of the angles of such a figure to the angle opposite, will, art. 69, sub. div. 1, 2, 3 and 4, divide it into two equal right angled triangles: whence we conclude that the area of the right angled triangle is represented by half the product of its sides.

Fig. 146.



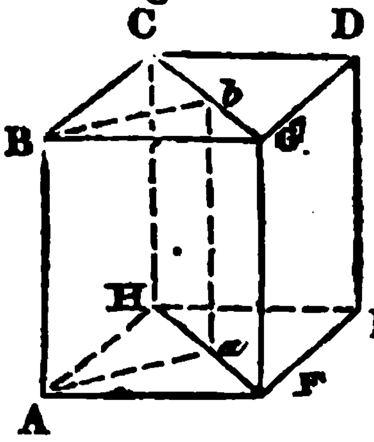
81. 1. By dividing three adjacent sides of a rectangular parallelopipedon into linear units, and drawing through the points of division planes parallel to the remaining sides, we shall, art. 75, decompose the solid into cubes each of whose sides is a linear unit; and

hence, art. 78, the number of solid units in the rectangular parallelopipedon will be equal to the number of these cubes; or to the product obtained by multiplying together factors which, respectively, represent the number of units in each of the adjacent sides.

Or, calling one of the faces of the solid *the base*, and the side perpendicular to this face *the altitude*; the solidity will be equal to the product of the base into the altitude.

81. 2. And dividing the parellelopipedon into triangular prisms and pyramids, we can readily obtain the solidity of these figures, which will again serve us as instruments to discover the solidities of figures more complex.

Fig. 147.



The decomposition into prisms may be conducted as follows.

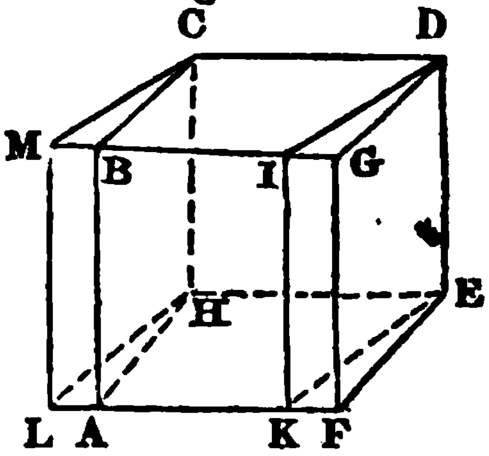
First. Passing a plane through the sides CH and GF, we divide the parallelopipedon into two equal triangular prisms; the solidity of either of which will, therefore, be equal to the product of its base into its altitude.

Secondly. Dividing the angle contained by the planes ABCH and ABGF into two arbitrary portions, by means of the plane AB*ba*; we can employ the process used in the latter part of art. 64 to show that the solid *b*BA*a*FG is to the solid CBAHFG, as the base *a*AF is to the base HAF: but the solids have a common altitude, and the measure of the last is its base into its altitude; hence, the measure of the first is also its base into its altitude.

81. 3. And since by adding to the rectangular par-

allelopipedon ABCDEF the right prism CMLA, and taking from the same solid an equal prism DIKF, we neither increase nor diminish its magnitude, the parallelopipedon LMCDIKE, produced by this addition and subtraction from that before mentioned, will also be equal to its base into its altitude.

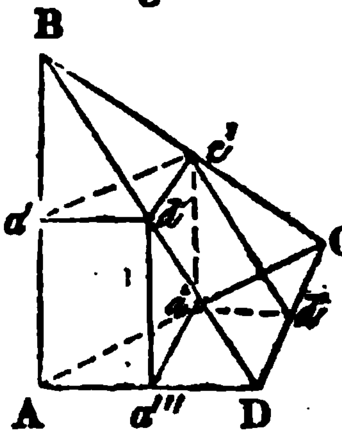
Fig. 148.



81. 4. And on similar principles we might now proceed to decompose the prisms we have thus obtained into rectangular pyramids; that is, into the species of solid we have chosen as our type; but the object of this inquiry is much better obtained by following a converse route, and decomposing a rectangular pyramid into parts that are either pyramids similar to itself, or right prisms.

To effect this decomposition, we have only to place at the angles B and C rectangular pyramids similar, latter part of art. 61, to that we are considering, but of half the linear dimensions; and, finally, to draw the lines $d'a'''$, $a''a'''$ parallel respectively to AB and DC.

Fig. 149.



The corresponding faces of the great and small pyramids in this figure, being at right angles to the same edges, are parallel, art. 70–2, and hence the given pyramid is decomposed into two similar pyramids, and two right prisms, $Aa'c'a''a'''$, $a'''a''c'd''D$, having triangular faces.

Now taking, according to our notation, page 109, $AB=a$, $AD=a'$, $CD=c$, we have the solidity of the first prism equal to half the product of its base into its altitude,

or, $\frac{1}{2}.\frac{1}{2}a.\frac{1}{2}a''.\frac{1}{2}c=\frac{1}{16}.aa''c$: and since the second prism stands on the base $Da''d'$, and has the altitude Dd'', its solidity will also be equal to $\frac{1}{16}aa''c$.

Whence the solidity of these two prismatic solids is $\frac{1}{8}aa''c$.

But this last is what remains after taking away from the pyramid ABCD the two smaller pyramids.

Again, the two smaller pyramids are identical, and, therefore, if r is the ratio which the solidity of one of these parts has to s, the whole pyramid; each of the smaller pyramids will be expressed by rs; and the sum of all the parts, or the solidity sought, will be expressed by

$$s=\frac{1}{8}aa''c+2rs. \quad . \quad . \quad . \quad (a)$$

But restricting the inquiry to solids of the same form, and recollecting that r is the ratio which the solidity of a rectangular pyramid bears to that of a second pyramid whose sides are double those of the first; we observe that r is either constant, or dependent on the *absolute* magnitude of the sides.

The last supposition, if not directly contrary to our previous inquiries, is, yet, little in accordance with them; and the rules of analysis will therefore direct us to regard, for the moment, the remaining hypothesis as the correct one, and to examine how far it aids our research, before we proceed to establish its truth.

But analysing either of the two smaller pyramids by the analysis that was used for the whole solid, and on the hypothesis adopted with regard to r, we shall obtain the equation

$$rs=\frac{1}{8}.\frac{a}{2}.\frac{a''}{2}.\frac{c}{2}+2r^2s. \quad . \quad . \quad . \quad (b)$$

And multiplying the equation (a) by r, and afterwards subtracting it from (b), there results

$$r = \frac{1}{8}.$$

Whence the equation (a), by subtraction and reduction, becomes

$$s = \frac{1}{6} a a'' c. \quad \ldots \ldots 10.$$

Or, which amounts to the same thing, the solidity is equal to the area of the base into one-third of the altitude; and as all pyramids can be resolved into rectangular pyramids placed side by side, art. 65, the solidity of every pyramid must be equal to the base into one-third of the altitude.

81. 5. To complete the investigation, it yet remains for us to establish the truth of the hypothesis assumed respecting r.

For this purpose divide each of the sides a, a', a'', of a rectangular pyramid, fig. 150, into the same number of equa parts, and through these division draw planes, respectively, perpendicular to the lines a, a', a'', o parallel to the planes A', $A_{,}$, A.

Fig. 150.
B

Any two consecutive section parallel to $A_{,}$ cut off a slice of th solid, and there are as many o these slices as parts in the side a'.

But the divisions parallel to A cut off similar portions of the solid, and of these portions the number is equal to the parts wherein a'' is divided.

The intersections of the second set of planes with the first set, lying in planes that never meet, will be parallel to each other, and hence, by art. 70—2, sub. 4, they will be perpendicular to A'.

Taking, then, two consecutive planes parallel to A, and two consecutive planes parallel to $A_{,}$, the intersections of these four planes will form the edges of a right prism *qmpn*, *q'm'p'n'*, whose base is a parallelogram, and whose opposite face lies in the plane B.

But if n is the number of parts into which each of the lines a, a', a'' is divided, and the lines D*h*, A*i* be, respectively, equal to m and to m' of those parts; we shall have

$$\frac{a'}{a''} = \frac{hl.}{\frac{m.a''}{n}} = hl.\frac{n}{ma''}$$

or

$$hl = \frac{ma'}{n}:$$

and

$$ql = \frac{ma'}{n} - \frac{m'a'}{n} = \frac{m-m'}{n}a'.$$

But

$$\frac{qq'}{ql} = \frac{a}{a'},$$

or

$$qq' = \frac{m-m'}{n}.a$$

Which being equal to an aliquot number of the parts, $\frac{a}{n}$, into which a is divided, we conclude that one of the

planes drawn parallel to A′ passes through q'. And by an investigation in all respects similar, the plane next superior to that we have considered will be found to pass through p' and m'; and the plane two degrees superior will be found to pass through n'.

The planes parallel to A′, but inferior to the $(m - m')$th, will evidently divide the column $pqmn$, $p'q'm'n'$ into $(m - m')$ right parallelopipedons, whose sides are $\frac{a}{n}$, $\frac{a'}{n}$ and $\frac{a''}{n}$.

The planes we have considered, not being restricted to any particular values of m, it is plain the whole pyramid will be divided into two solids, an inscribed solid composed of integral parallelopipedons, and a solid composed of broken parallelopipeds that are bounded by the plane B.

But repeating the same operations on a similar pyramid, whose sides we will denote by α, α' and α'', and retaining n the same, the inscribed solid in this last pyramid will be composed of the same number of parallelopipedons as the inscribed solid in the former; but the sides of the parallelopipedons will here be $\frac{\alpha}{n}, \frac{\alpha'}{n}, \frac{\alpha''}{n}$.

The solidity of a parallelopiped in the first solid, will therefore be to the solidity of a parallelopiped in the second, as a^3 to α^3, and it is plain the inscribed solids will be in the same proportion.

Now, calling the two pyramids P and P′, their solidities S and S′, and that of their inscribed solids s and s'; if we have not $\frac{\mathrm{S}}{\mathrm{S}'} = \frac{a^3}{\alpha^3}$, let us take a third pyramid, P″, similar to the former, and let us denote its sides by b, b',

Fig. 151.

b'', its solidity by S'', and that of its inscribed solid by s''; and, since the magnitude of this pyramid is arbitrary, let us make

$$\frac{S}{S''}=\frac{a^3}{a^3}$$

and suppose, first, that S'' is less than S'.

D

The number n may be taken so great that $b-a$ shall be greater than $\frac{2a}{n}$, $b'-a'$, greater than $\frac{2a'}{n}$, and $b''-a''$ greater than $\frac{2a''}{n}$.

But superposing the pyramids P' and P'', and causing the angles at A, to coincide; the angles BDA, EGA, which measure, respectively, the inclination of B with A', and E with A', will be equal, and the planes B and E parallel.

Now supposing $n''p''q'm''$ to represent the section of the prism which is made by a plane parallel to A', and which passes through q, the lines $n'n''$, as we have seen in the preceding demonstration, will be equal to $\frac{2a}{n}$, and the lines $p'p''$ and $m'm''$ to $\frac{a}{n}$.*

But the portions of any line parallel to a that is intercepted between the two planes must equal $b-a$, and exceed $\frac{2a}{n}$.

Hence $n''p''q'm''$ falls wholly without EGF; and the solid which is inscribed in P' will exceed the pyramid P''.

* Some of the small letters are omitted in the figure.

Moreover it has been shown that

$$\frac{s}{s'}=\frac{a^3}{\alpha^3};$$

and $s < \mathrm{S}$, and $s' > \mathrm{S}''$, and, since to increase the numerator and decrease the denominator of a fraction manifestly alters its value, the equation

$$\frac{\mathrm{S}}{\mathrm{S}''}=\frac{a^3}{\alpha^3}$$

is shown, on the hypothesis we have adopted, to involve an absurdity.

S″, therefore, cannot be less than S′; but neither can it be greater.

For in that case we should have $\frac{\mathrm{S}}{\mathrm{S}'}$ a greater number than $\frac{a^3}{\alpha^3}$, and consequently $\frac{1}{\frac{\mathrm{S}}{\mathrm{S}'}}$ or $\frac{\mathrm{S}'}{\mathrm{S}}$ a less number than $\frac{1}{\frac{a^3}{\alpha^3}}$ or $\frac{\alpha^3}{a^3}$; and, S‴ representing the solidity of a pyramid, P″, less than P, we might have

$$\frac{\mathrm{S}}{\mathrm{S}'''}=\frac{a^3}{\alpha^3}$$

an equation which could be proved to be absurd by proceeding with the pyramids P and P‴ precisely as we proceeded with P′ and P″.

Since then S″ can neither be less nor greater than S′, it must be equal to it; and we have the equation

$$\frac{\mathrm{S}}{\mathrm{S}'}=\frac{a^3}{\alpha^3}.$$

In the particular case that we sought to demonstrate,

$\frac{S}{S'}$ was put equal to r, and a was $\frac{1}{2}$ of α; making which substitutions in the equation just derived, we obtain

$$r = \tfrac{1}{8};$$

a *constant* value, which is independent of the *absolute* magnitude of a.

81. 6. The analysis used in this demonstration deserves attention. It is the simplest form of a method, the method of limits, extensively used in mathematics, and that in comparing quantities, A and B, not admitting of direct comparison, proceeds, as above, by dividing them into two parts, having the following properties,

1. The greater part of A is such as to admit a direct comparison with the greater part of B.

2. The remaining parts, whether of A or B, are quantities capable of being rendered as small as the nature of the problem requires.

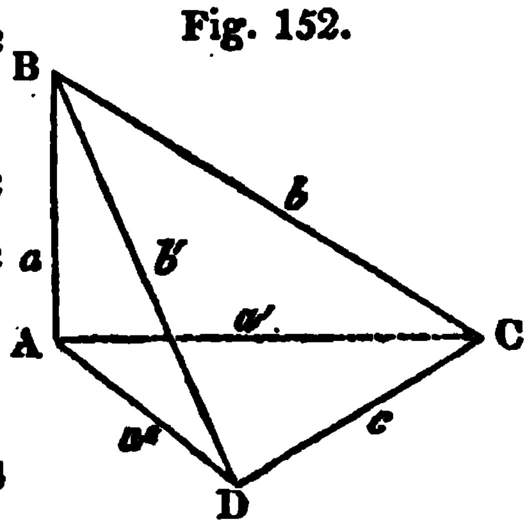
The *elementary* parts used both in the preceding problem and in art. 49, are quantities of the latter kind; and, in general, if we divide, as far as possible, any quantity A into such elementary portions, the sum of the integral elements will constitute the *first* of the two parts above described, as the sum of the divided or broken elements will constitute the *second*.

82. 1. At the close of the 79th art. we proposed reconsidering the proposition of art. 74, with the view of extending its application, until this included not only the linear relations of a definite number of points, but, also, their relations of solidity and superficies: the measures that we have since obtained, of the area of a right

angled triangle, and the solidities of right prisms, and rectangular pyramids, apply immediately to this investigation; and will readily lead to the extension required.

The analysis of solids pointed out in art. 65, sufficiently indicates the route we are to take, and requires us to commence the investigation with the relation which exists between the oblique face and the base of a rectangular pyramid.

Fig. 152.



The areas of those two surfaces are, by art. 80—2, $\frac{b'c}{2}$ and $\frac{a''c}{2}$; and since $a'' = b'$ cos. $b'a''$, or, which is the same thing, b' cos. BA′;

$$A' = \frac{b'c}{2} \cos.\ BA'$$

$$B = \frac{b'c}{2}$$

or

$$A' = B \cos.\ BA' \quad . \ . \ . \ . \ . \ . \quad (11)$$

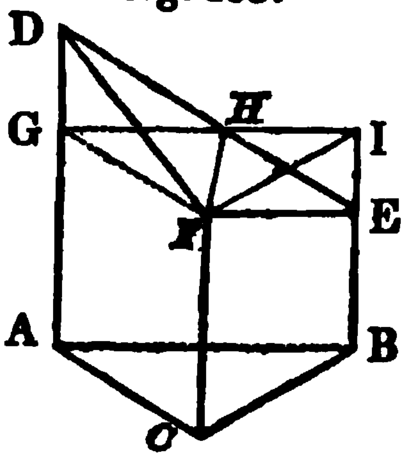
where A, B and A′ are not used to denote the whole of those infinite planes, but merely the finite portions of them which form the faces of the pyramid.

82. 2. But although in the decomposition of pyramids we need no other figure than the rectangular pyramid, the case is otherwise with solids of a more complicated nature; in decomposing which, an additional instrument is required; an instrument that shall answer in this investigation, the purpose to which parallel lines and planes were applied in the decomposition of linear figures.

The right prism, the additional solid required, is indeed formed by parallel lines and planes, and serves merely as a basis whereon to place the rectangular pyramids used in the process of decomposition.

Let us assume a right polyedron ABCDEF, fig. 153, whose base is a triangle, and through F, that angle of its superior surface which is neither the most, nor the least distant from the base, pass a plane, FGHI, parallel to the latter; we shall then obtain an easy case of the method of decomposition alluded to. For producing the planes A'' and B, until they meet the plane F, we shall form, in this way, a right prism AGIBC, on the superior surface of which, though in contrary directions, are placed the triangular pyramids DGFH, and FHIE.

Fig. 153.



These last are not necessarily rectangular, but every triangular pyramid which is not rectangular, may be decomposed into two rectangular pyramids, placed side by side, and therefore we shall have,

$$F = D' \cos. D'F$$
$$F' = E \cos. EF'$$

or, observing that $ABC = IFG = F + F'$, and denoting DFE (as one plane) by D,

$$A = D \cos. DA, \quad . \quad . \quad . \quad . \quad (12)$$

82. 3. The theory of triangles given in Part III. and which is merely a development of the principles already established, gives,* when applied to the solid we are considering,

* Note 3.

$$\mathrm{F} = \tfrac{1}{2} f'g \sin. f'g$$
$$\mathrm{F}' = \tfrac{1}{2} f'h \sin. f'h$$
$$\mathrm{F} + \mathrm{F}' = \mathrm{A}.$$

But putting S′ for the solidity of the whole polyedron ABCDEF, we shall also have,

$$\begin{aligned}
\mathrm{S} &= \mathrm{A}.c'' + \tfrac{1}{3}\left\{\mathrm{F}.\ (a'' - c'') - \mathrm{F}'\ (c'' - b'')\right\} \\
&= \mathrm{A}c'' + \tfrac{1}{3}\left\{\mathrm{F}a'' - \mathrm{A}c'' + \mathrm{F}'b''\right\} \\
&= \tfrac{2}{3}\,\mathrm{A}c'' + \tfrac{1}{3}\left\{\mathrm{F}a'' + \mathrm{F}'b''\right\} \\
&= \tfrac{1}{3}\,\mathrm{A}c'' + \tfrac{1}{3}\,\mathrm{A}a'' + \tfrac{1}{3}\,\mathrm{A}b'' + \tfrac{1}{3}\,\mathrm{A}c'' - \tfrac{1}{3}\,\mathrm{F}'a'' - \tfrac{1}{3}\,\mathrm{F}b'' \\
&= \tfrac{1}{3}\,\mathrm{A}\left\{a'' + b'' + c''\right\} + \tfrac{1}{3}\,\mathrm{A}c'' - \tfrac{1}{3}\,\mathrm{F}'a'' - \tfrac{1}{3}\,\mathrm{F}b'' \\
&= \tfrac{1}{3}\,\mathrm{A}\left\{a'' + b'' + c''\right\} + \tfrac{1}{3}\,\mathrm{A}c'' - \tfrac{1}{3}\,\mathrm{F}'c'' - \tfrac{1}{3}\,\mathrm{F}c'' \\
&\qquad + \tfrac{1}{3}\,\mathrm{F}'\,(c'' - a'') + \tfrac{1}{3}\,\mathrm{F}\,(c'' - b'') \\
&= \tfrac{1}{3}\,\mathrm{A}.\left\{a'' + b'' + c''\right\} + \tfrac{1}{3}\,\mathrm{F}\,(c'' - b'') - \tfrac{1}{3}\,\mathrm{F}'\,(a'' - c'') \\
&= \tfrac{1}{3}\,\mathrm{A}.\left\{a'' + b'' + c''\right\} + \tfrac{1}{6} f'g \sin. f'g\,(c'' - b'') - \tfrac{1}{6} \\
&\qquad f'h \sin. f'h\,(a'' - c'')
\end{aligned}$$

And from similar triangles,

$$\frac{c'' - b''}{h} = \frac{a'' - c''}{g}$$

or

$$g\,(c'' - b'') = h\,(a'' - c'')$$

and

$$\mathrm{S} = \tfrac{1}{3}\,\mathrm{A}\,(a'' + b'' + c'')$$

82. 4. The solid ABCDEF, which we have decomposed into a right triangular prism, and four rectangular pyramids, is the only species of solid that will be required in the decomposition of the most complicated polyedrons.

The process will nearly resemble that used for linear figures: and will be readily understood from the following example.

Sect. VI. Of the different species of quantity which are included in the relations of a finite number of points.

Art. 82. The general analysis of figures extended to their relations of area and solidity.

Fig. 154.

Let ABCDEF represent a so-
lid whose surface consists of man
triangular faces: from each of i
angles let fall perpendicula
upon a plane, M, assumed
pleasure: and connect all t
points, *a*, *b*, *c*, &c. in which t
perpendiculars meet the plane

The figure traced upon the latter may consist of many polygons, but it is evident that one of these figures will enclose the others, as is the case in the present instance with the figure *abef*.

The several lines traced in the plane M, are known as *projections* of those edges to which they correspond in the solid.

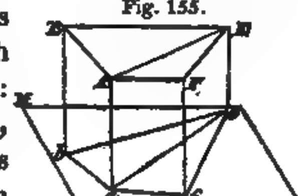
Thus, the figure *abef* is the "projection" of ABEF.

And this last may be regarded as a boundary separating the faces of the solids into two sets: namely, a set, each face of which is more remote from the plane than is the boundary in question; and a set that lies between the boundary and the plane.

In the example we have chosen, the second set consists of a single face, $A_{,,}$.

But passing a plane through the lines A*a*, E*e*, which, being perpendicular to M, are parallel, the intersections of this plane with $A_{,,}$, and its projection *befa*, will divide each of those areas into two triangles: and as the solids ABE*a*, AEF*a*, to which this construction gives rise, are right polyedrons with

Fig. 155.



triangular bases, the equation 12, art. 82—2, will apply to them, and produce results that, added together, give

$$abef = A_{,,} \cos. A_{,,}M.$$

But the polygon *abef*, fig. 155, which is the projection of $A_{,,}$, a single face of the solid, is seen in figure 154 to be composed of the projections of all the other faces; and as any one of these projections, *abc*, for example, forms the base of a right polyedron with a triangular face, we have the following equations:

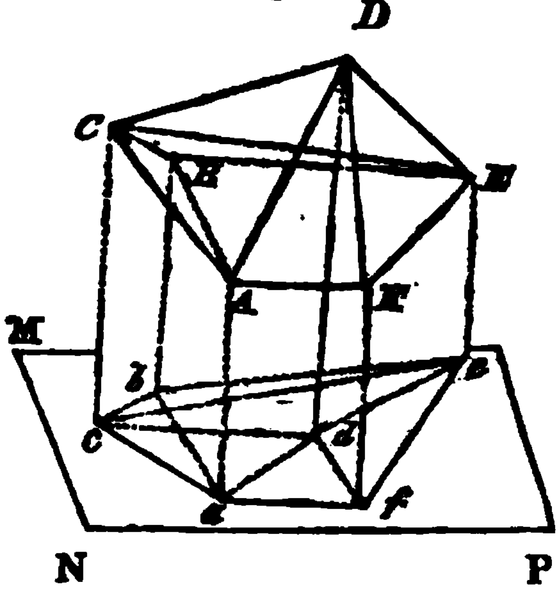
$$A_{,,} \cos. A_{,,}M = A \cos. AM + A' \cos. A'M + A_{,}'' \cos. A_{,}''M + B_{,} \cos. B_{,}M + C \cos. CM + D \cos. DM, \ldots 13$$

a formula perfectly similar to that in art. 74, and, in like manner, capable of being extended to all cases; a truth that we shall readily perceive by examining the changes that take place when those faces which we have denominated the second set, no longer lie in one plane.

The perpendiculars, or some of them, might in this last case fall without the line *abef*, which would then cease to be a boundary.

Let us suppose, for example, the solid to take the form shown in fig. 156, where the boundary that separates the two sets of faces is ACBEF, and the second set, instead of the single area $A_{,,}$, consists of the two areas $A_{,,}$ and A.

Fig. 156.



The projection *acbef* is here also composed of the sum of the projections of the areas superior to ACEF, areas which we have denominated the first

set; but to obtain, as before, the projection of $A_{,,}$, from the sum here mentioned, we must subtract, in the instance before us, the projection of A, and in general, the projection of every inferior area except $A_{,,}$.

And hence, for the equation 13 to apply to fig. 156, it is only necessary for the term A cos. M to become negative.

Now this will take place; for a plane meeting a second plane makes with it adjacent angles that are supplementary; and as the cosine of the supplement differs from the cosine of the angle itself merely in having a negative sign, the factor cos. A*m*, and, consequently, the term A cos. A*m*, will become negative, as required; provided, however, the inclination of A to M is measured by the greater of the two angles which those planes form.

In the most general case, the rule we have here obtained with respect to the inclination of A, must be applied to all the inferior areas that enter the left hand number of 13, which then becomes perfectly general.

82. 5. An equation not very dissimilar to that we have investigated, expresses the solidity of the polyedron. For this solidity is, fig. 154, dependent on that of the right polyedrons constructed on the plane M, and whose bases are in that plane; and is obtained by subtracting from the solidity of the right polyedrons whose upper surfaces are the first set of areas, that of the right polyedrons whose upper surfaces form the areas of the second set.

To deduce from this result the equation in question, denote the perpendiculars let fall on M by the letter *p*,

distinguishing them apart by accenting p according to the following rule.

"Observing that each perpendicular is drawn from an angle of the solid, and that every angle is denoted by a letter having a certain rank,

"Place over p as many accents as there are units in the rank of the letter whence the perpendicular is drawn."

The equation which expresses the solidity required will, then, readily follow from this reasoning; for, according to the rule we have here given, the solidity of the polyedron *abc*ABC, fig. 154, is

$$\tfrac{1}{3} \text{ A cos. AM } \{p' + p'' + p''\}$$

And a similar result being also true of the other polyedrons, we have

$$S = \tfrac{1}{3} \text{ A cos. AM } \{p' + p'' + p'''\} + \tfrac{1}{3} \text{ A}' \text{ cos. A}'\text{M } \{p' + p'' + p'''\} + \&c. \quad . \ . \ . \ . \ . \ 14$$

a formula in which, to permit the necessary changes of sign, we must adhere to the rules demonstrated in art. 82—4, respecting the inclination of the inferior areas.

CHAPTER II.

OF THE ELEMENTS TO WHICH PLACE IS REFERRED.

PRELIMINARY REFLECTIONS.

Hitherto we have considered each problem as distinct, but all geometrical propositions can only be parts of an infinite series of relations connecting all the points in space—these relations resulting from the positions of the points, it follows, that if we could express and tabulate the position of all points in space, such a table would implicitly contain their relations.

But how is such a table to be formed—for space being uniform and infinite, place is only relative?—*Place*, then, *must be rendered absolute*, by assuming some fixed point to which all others shall be referred.

But the references to this point will be, first, distance; and secondly, directions according to which the distances are measured.

It will be necessary, then, not only to assign a point whence our measurements shall commence, but also directions according to which they shall be reckoned.

INQUIRIES SUGGESTED BY THESE REFLECTIONS.

To how many invariable or *primordial* elements must we refer the position of a point in order to distinguish it from any other point in space?

Calling the measurements taken with reference to these primordial elements, co-ordinates, how shall we express the relations of points in terms of their co-ordinates?

SECTION I.

VARIOUS SYSTEMS OF PRIMORDIAL ELEMENTS.

Data involved in the position of a point—position of a point merely relative, and determined by relation to primordial elements—relations of a point to the primordial elements expressed by co-ordinates—examination of the cases that occur when the co-ordinates are rectangular—these various methods of expressing place depend on three primordial elements and three co-ordinates—further examination of the cases that occur—oblique co-ordinates—names of the primordial elements and of the co-ordinates—distinction between linear and polar co-ordinates—the parts of an open polygon assign the position of a point—subject of art. 88 *continued—method of projections—perspective representations—remarks concerning different cases which occur in the method of projections.*

83. According to the views developed in the preceding pages, geometry is the science of position; and as points are the emblems of place, and comparative place is only another term for position, every proposition in geometry must be an inquiry or an assertion respecting the relations of points.

and z are known; for only one perpendicular can be drawn through P to the given plane; and from B, the foot of this perpendicular, only one perpendicular, BA, can be drawn to the given line OX.

85. The three lines, x, y and z, are termed the *rectangular co-ordinates* of the point P. And, in general, whatever are the primordial elements, the measures, whether lines or angles, whereby the position of a point is referred to them, are termed the *co-ordinates* of that point.

It is at once evident that any elements which suffice to determine the solid OABP are also sufficient to fix the position of the point P. But as the equations 2, 3 and 9 reduce the investigation of this solid to a known problem, the solution of a given number of equations, we may regard its properties as rigidly demonstrated; now it results from these equations, as we shall hereafter show, Part III.,* that only three elements of the solid in question can be regarded as independent; and, consequently, the position of P will always depend on three co-ordinates.

86. The choice of these is not however perfectly arbitrary, but is subject to two conditions; first, that one, at least, of the co-ordinates should not be an angle; and secondly, that all three of the co-ordinates should not belong to one face of the solid.

Of the chief cases wherein these conditions are fulfilled, we need only mention, here, the two following:

* Note 4.

1. When the given things are the three rectangular co-ordinates x, y and z.

2. When the given things are the distances r, and the angles (rx), (sy).

Other cases that it will be necessary to notice, might, indeed, be derived in this way; but they have a more immediate reference to the relations of four points, and accordingly, from that source we shall obtain them.

Fig. 159.

B

A

D

The relations of four points, it has already been remarked, constitute a triangular pyramid; and by letting fall from B a perpendicular on the opposite face, we can decompose this solid into six rectangular pyramids, arranged as in the diagram. Moreover, by applying to each of these the arguments used respecting the rectangular pyramid of the preceding article, we may show that only one triangular pyramid can be so placed on the base ACD as to have its vertex at B. If, then, the base ACD is given, whatever elements suffice to determine B, will also suffice to determine the whole solid, and conversely, whatever elements determine the solid, will determine B.

But since each of the triangular faces about the point B contains a side of the known base, it is only necessary that we should know two other elements in each of these triangles, art. 51, in order completely to determine them.

Or, restricting the data still further, we finally perceive that a knowledge of such elements in any two of the triangles here mentioned will suffice: since, adding to these triangles the known base, we should have all three sides of the remaining triangle—data, from which that triangle can be found.

Chap. II. Of the elements to which place is referred.
Art. 86. Examination of the cases that occur. Art. 87. These various methods of expressing place depend on three primordial elements and three co-ordinates.

These considerations would enable us to add several other systems of co-ordinates to those (1 and 2) enumerated in the preceding article; but the only cases that need be mentioned, are,

3. When the three given things are the distances from three known points.

4. When the three given things are the distance from a known point, and the angles formed by each of two given straight lines with the direction constituting that distance.

87. Examining with attention these various methods of expressing *place*, we shall find them all to depend on similar data, on *three primordial elements, and three co-ordinates*.

The primordial elements may be points, or lines, or surfaces;—the co-ordinates, lines, or surfaces, or angles.

88. What is here said would be sufficient to complete the theory we are considering; but the subject is of too much importance to be hastily dismissed, and there is another method of viewing it that deserves our attention.

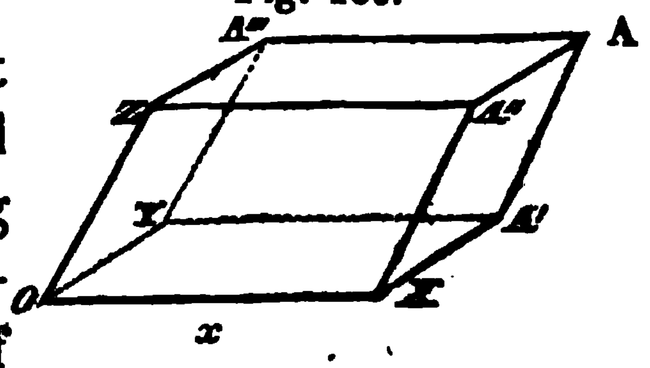
In what preceded we considered points as the symbols of place, and lines and surfaces as their relations: but it is often advantageous to reverse the process, and to regard lines and points as formed by the intersection of given surfaces.

According to this arrangement, a straight line would be regarded as the intersection of two planes; and a point—the symbol of place—as the intersection, either, of three planes—or two straight lines—or of a straight line and a plane.

The object of course in these cases, as in the cases that have preceded, is to refer the position of many elements to those of a few assumed as known: and, accordingly, when we speak of given planes we should be understood as meaning planes whose position is assigned with respect to known primordial elements. The nature and choice of these have been already explained, but in considering the first among those methods of fixing a point which we have now in view, the primordial elements should be three planes.

89. Let us assume for this purpose the planes which meet at O. The position of all planes parallel to these and at a given distance from them will be assigned. And drawing three such at the assumed distances x, y, z; A, the point of their intersection, will be completely determined, and the six planes, together, will enclose a parallelopiped.

Fig. 160.



The distances x, y and z, were measured along the common intersections of the primordial planes, and since these might have been assumed as forming any required angles with each other, we have a right to suppose the angles formed by x, y and z as equally arbitrary.

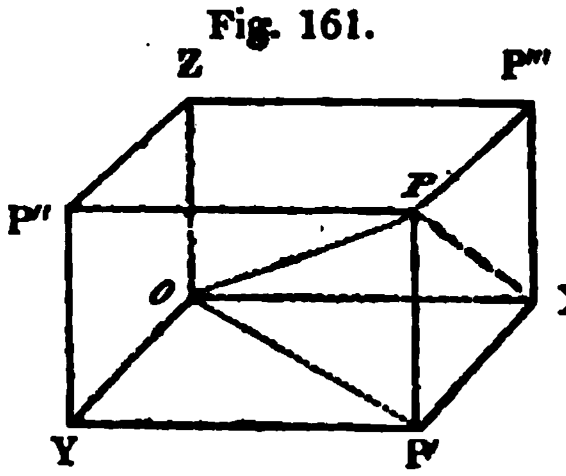
But the lines AA′, AA″, AA‴ are equal and parallel to x, y and z; and thus we may suppose the position of A, assigned by its distances measured in known directions, from three primordial planes.

These distances are the co-ordinates of A, and may, also, be measured by the lines OX, XA′, A′A, which are equal and parallel to the former.

Chap. II. Of the elements to which place is referred.
Art. 89. Oblique co-ordinates. Art. 90. Names of the primordial elements and of the co-ordinates.

When the primordial planes are rectangular, this method of determining a point perfectly agrees with that in art. 84; and, in fact, drawing OP, OP′ and PX, we form the rectangular pyramid employed in that article.

Fig. 161.



90. When the primordial planes are not rectangular, the system is termed that of *the oblique co-ordinates;* and we observe that what is said in case 1, art. 86, should be understood of ordinates of this kind, and not merely of those which are rectangular.

The point O, fig. 158 and 160, is termed the *origin*.

The directions OX, OY, OZ, the *axes of the co-ordinates;* the first is termed the axe of the *x's*, the second the axe of the *y's*, the third the axe of the *z's*.

The planes passing through O are termed indifferently the *primordial*, or the *co-ordinate planes*. Of these, the plane passing through the axe of the *x's* and the axe of the *y's* is termed the plane of the *xy's;* as those passing through the axes of the *x's* and *z's*, and of the *y's* and *z's* are, respectively, termed *the planes of the xz's* and *the yz's*.

91. Reviewing what has been here said we remark, that, of the cases, 1, 2, 3 and 4, which we have enumerated, the first and third assign the position of a point altogether by *linear co-ordinates;* whilst the second and fourth assign its position by co-ordinates that involve a line and two angles. This line, or distance, is called the *radius vector*, and as the origin from which it is reckoned

has received the name of the *pole*, the two methods alluded to have, on this account, been designated "the methods by *polar co-ordinates*."

92. Each of the four systems is expressed, art. 86, in parts of the same closed figure, and hence, by means of the equations derived from the latter, the co-ordinates of a point whose position is assigned by one of the four systems, may be transformed into co-ordinates having reference to either of the other three. Such equations are called "equations for transforming the co-ordinates," and will be treated of in a subsequent section; but in the mean time we may remark, that,

Every system of primordial elements assigns the position of a point by giving the parts of an open polygon, having the origin at one of its extremities, and the point to be assigned at the other.

And further, that all the equations which can be deduced from the relations of a point so assigned, are obtained by closing the polygon in question.

93. As the processes described in the first part of this section for assigning the position of a point in space, are perfectly adequate to that purpose, it would appear unnecessary to pursue in detail the views suggested in the 88th art. or to embarrass the student with a second method of accomplishing objects which he has already learned to attain. But the riches and resources of analysis often depend upon these various ways of assigning a point—and, far from being useless, the proposition before us is of continued application; a single case of it—the only

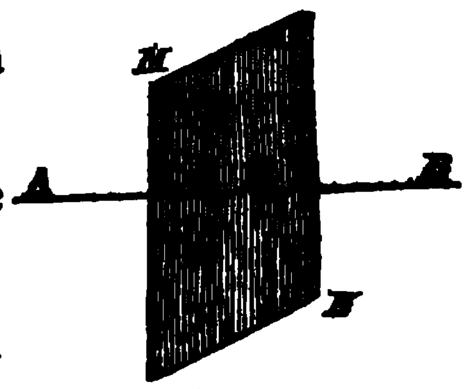
remaining one which we need consider, forming the basis of several entire branches of science.

This case, the last of the methods enumerated in the 88th art. is that wherein a point is regarded as the intersection of a straight line and a plane.

To assign a point by the intersection of a line with a plane, it is of course necessary, that both the line and plane should be given. A straight line is given when we know two points through which it passes, and in the first consideration of the problem before us we shall regard the straight line as determined in this way.

Suppose, to illustrate the subject by reference to an example, AB and MN the given line and plane, whose point of intersection is P.

Fig. 162.

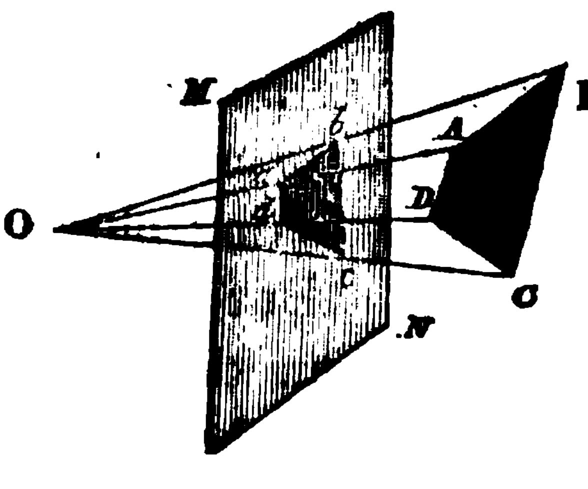


Since MN and AB are given, the point of intersection is assigned: and, conversely, if P is known, a line drawn through A and P will be a line given in position.

The first of these cases need not be considered, but the second, or inverse case, is that which we proposed to discuss.

94. To illustrate its application, let us supppose ABCD a polygon whose sides are not in one plane. Assume the point O, and the plane MN as primordial elements; and from O to each point of the polygon draw the lines OA, OB, OC, OD.

Fig. 163.



Then, since the plane MN is indefinite in extent, the lines in question, or those lines produced, will each meet this plane in a point; and joining a, b, c, d, the points

where OA, OB, &c. meet MN, we shall form on the plane an image, or picture of the polygon.

This image, it is true, will not be a figure similar to the polygon it represents; and, in fact, an infinity of polygons may all be represented by the same picture; the polygons ABCD, A′B′C′D′, for example, are each represented on the plane MN by the picture *abcd*. Nevertheless, this image, or picture, is connected with its object by laws which it is the business of analysis to discover; laws that will be better explained in another place, since our inquiries in this section, confined to the methods of determining points, do not extend to the relations deduced from them.

Fig. 164.

Now when, together with the image of the polygon, we have given the distances of the points, A, B, C, D, from their representations on the plane, the end proposed is accomplished; since a knowledge of the figure *a*, *b*, *c*, *d*, together with a knowledge of the distances *a*A, *b*B, *c*C, *d*D, will afford data that enable us to assign the position of A, B, C and D.

This particular case of the problem, that requires us to assign the position of a point in space, is known in geometry as the method of *projections*; and the figure *abcd*, the image or picture of ABCD, is termed the *projection* of the latter.

Although, as we have remarked, the theory of projections is only a particular case of the more extended problem we are considering in this section, it is yet a case so extensive, that its subdivisions constitute separate branches of science.

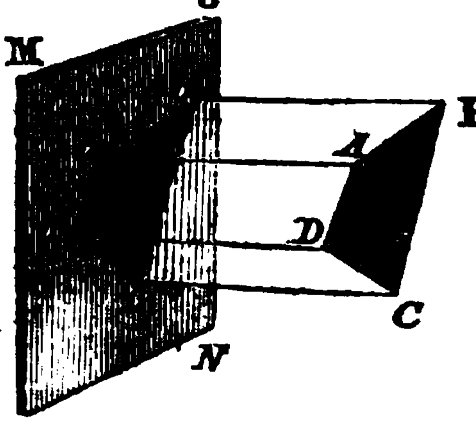
95. When, for example, the projection is determined, as in the figure 163, with reference to a fixed point O, and to a single plane MN; and when, at the same time, no account is made of the distances aA, bB, &c.; the method is designated as the theory of *perspective;* and the projection *abcd* is termed the *perspective representation* of the *object* ABCD.

The latter is here considered as a sensible existence, viewed by a spectator whose eye is placed at O: the plane MN as a transparent screen, placed between the eye and the object it views, and the perspective representation as a visible picture of the object.

To illustrate what is here said more fully, let us imagine an object placed beyond a window, or a perfectly smooth plate of glass, to be contemplated by an eye situated on the other side. Let us also imagine lines drawn from the eye to every point in the object; and suppose further, that wherever these lines meet the plate of glass, or the window, a very fine dot is made, to mark the point of intersection; the assemblage of all these points will constitute precisely the picture, or perspective representation we proposed to illustrate.

When we continue to increase the distance of the point O, both from the object and the plane, the theory of parallel lines, art. 69, will convince us that OA, OB, &c. become more nearly parallel: and, consequently, if from A, B, C, D, we draw parallel lines, and produce them until they meet the plane MN, in the points, a, b, c, d, the figure so formed will differ but little from the pictures we have described as perspective representations.

Fig. 165.



Indeed, if we consider that O, fig. 163, may be taken so distant as to make the angle BOC smaller than any angle that can be assumed, or, which amounts to the same, to make the sum of the angles OBC, OCB, as near as we please to $\frac{1}{2}$; it will follow that BO, and OC, may be brought within any assignable limit of being parallel. It is in this sense that *abcd*, formed by drawing parallel lines from the object to the plane, is said to coincide with a perspective representation, namely, with the picture the eye would perceive if we could suppose it at an infinite distance.

The *projecting lines*, A*a*, B*b*, &c. are usually drawn at right angles to the plane whereon the figure is projected; and the picture *abcd* is then called the *orthographic projection* of ABCD.

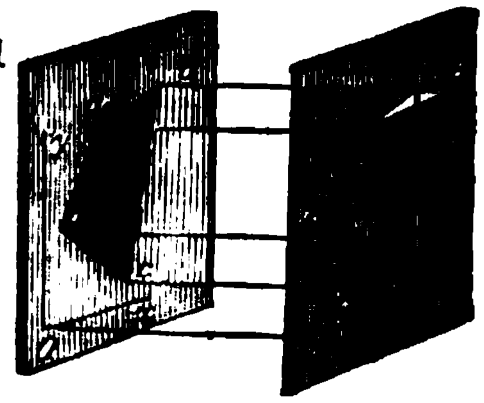
96. Although in describing the method of projections we have mentioned only two primordial elements, the reader cannot fail having noticed that here, as elsewhere, three such elements were really used. We Fig. 166. found, it is true, the points A, B, C and D, by means of *a*, *b*, *c*, &c. but these latter, when assigned analytically, that is, when assigned by means of symbols, and without the use of an accurately constructed diagram, must be referred to primordial elements.

Using for this purpose two rectangular axes Ox and Oy, we may assign the position of any point in the figure, as *a*, by means of its distances from these axes, namely,

Chap. II. Of the elements to which place is referred.
Art. 96. Different cases which occur in the theory of projections.

the distances Oa' and $a'a$; which, together with the distance Aa, enable us to determine the position of A. But drawing through A a plane parallel to MN, the parallels aA and Oz intercepted between these planes are equal, and thus the method of determining A by its projections on MN, and its distance from that plane, is seen to be identical with the method of co-ordinates, and to depend on the three primordial axes Ox, Oy, Oz.

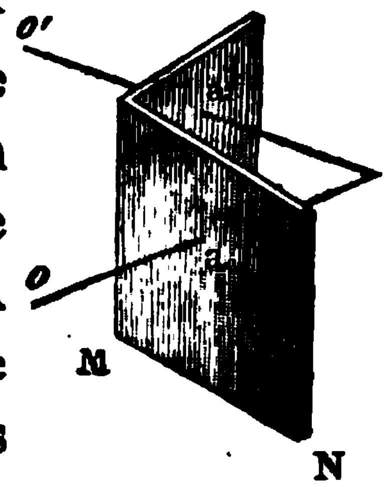
Fig. 167.



Another subdivision of the case we are considering, and which does not require any distances to be given, is that wherein a point is determined by its projections on two planes.

If we assume, for example, O and O′, and the planes MN and MN′ as primordial elements, and we have given the points, a, a', wherein lines drawn from O and O′ to any point A intersect the primordial planes, it is evident that A will be completely determined; since we have only to suffer the given points O and a, O′ and a' to be joined by Oa, and $O'a'$, and these lines produced will meet in the point A required.

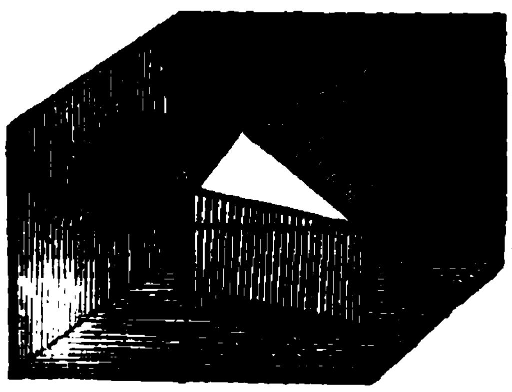
Fig. 168.



And thus, if we had two projections, or perspective representations, m and n, of any given figure ABC, the figure, as well as its position, would be completely assigned.

And indeed, when the projection is orthographic, this method of determining a point agrees with

Fig. 169.



those already described in arts 84 and 89, or may be made to do so by the addition of a third plane OXY.

The method of art. 84 requires the planes to be mutually rectangular, and, whenever the nature of our inquiry permits it, this will always be found the most convenient assumption; but whatever is the mutual relation of the planes, we observe that any two of the projections m, n or p, will be data sufficient to determine the third, since they suffice to assign the figure projected.

The orthogonal projections that have occupied us in the latter part of this section, are among the earliest improvements of practical geometry, but they originated with the necessities of the artist, and were unknown to the analyst until MACLAURIN deduced from them the principle of the three rectangular co-ordinates, or primordial elements, developed in art. 84, and which has since been employed as one of the most powerful instruments of geometrical reasoning.

With the artist such pictures conveniently supersede the necessity of models; and when the projections of an object, whether a building, a piece of carpentry, or a machine, have been accurately delineated, they may be reduced according to some fixed scale; and then drawings, of a few inches in size, will still contain so perfect a record of the object represented, that a similar one could at any time be executed from them.

Chap. II. Of the elements to which place is referred.

Art. 97. Geometry depends on the distance between two given points, and the angle formed by two given lines.

SECTION II.

THE ELEMENTARY RELATIONS OF POINTS EXPRESSED IN TERMS OF THEIR CO-ORDINATES.

Geometry depends on two elements, the distance between two given points, and the angle formed by two given lines—theorem for shifting the origin—distance between two points expressed in terms of their linear co-ordinates—angle formed by two given lines expressed in terms of the elements that assign their direction—theorems for transforming the co-ordinates.

97. As all our notions of linear geometry have been deduced, in the preceding pages, from the relations of points, it follows that every problem concerning rectilinear figures, either treats of such relations, or can, by a proper analysis, be reduced to them.

But the ultimate relations of points are the distances between the points, or the angles formed by their directions.

Sect. II. The elementary relations of points expressed in terms of their co-ordinates.

Art. 97. Geometry depends on the distance between two given points, and the angle formed by two given lines.

It is true, a plane passing through three points is a relation of them; but the area of a plane surface depends on the position of the lines that bound it; and the inclination of planes is measured by an angle between lines; and thus, in its final analysis, the geometry of a determinate number of points must reduce itself to expressions for distances that are sought, and angles formed, either by those distances, or others that are given.

Hereafter, a more extended examination will show us that, not only the relations of a given number of points, but, with certain restrictions, the whole of geometry may be reduced to these two elements; and, accordingly, it becomes essential to express, in terms of the various systems of co-ordinates explained in the preceding section, the distance between two given points, and the angle formed by two given lines.

By *a given point* we understand, in analytical geometry, *a point whose co-ordinates are given.*

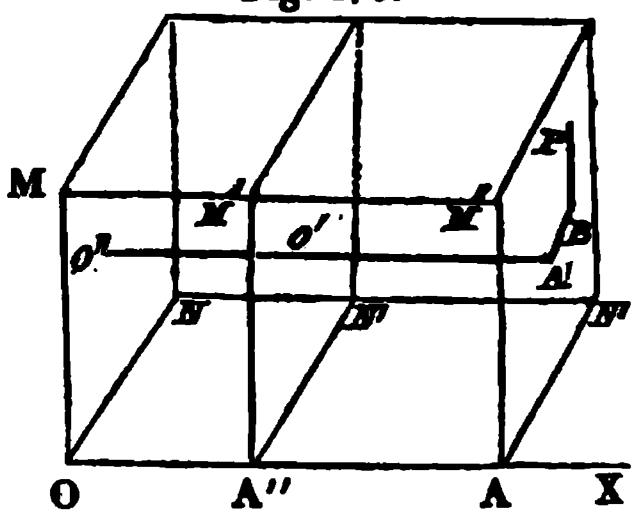
The problem before us will, therefore, vary with the system of co-ordinates employed, and requires a separate solution for each. In conducting our investigation, we shall commence with the system that assigns the position of points by linear co-ordinates, or, in other words, by distances measured in given directions, and from a common origin.

Our inquiry is, in this case, restricted to the discovery of formulæ which shall express, in terms of such co-ordinates, the distance between two points, and the angle between two lines.

It subdivides itself into several distinct problems, of which the first may be expressed as follows:

98. *The rectangular co-ordinates of a point being reckoned from a given origin, to find its co-ordinates when the origin is varied.*

Fig. 170.



Let planes MN, M′N′, M″N″ be drawn through the former origin O, the new origin O′, and the given point P; and let each of these planes be perpendicular to OX, one of the axes.

The lines OA and O″A are equal, (see art. 70—5), as also the lines O″O′, OA″ and likewise O′A′, A″A.

But OA″ will be an ordinate of the new origin O′, and OA, O′A′ will each be ordinates of the point P, the former measured from the primitive, and the latter from the new origin.

Let us denote OA″ by α, OA by x, and O′A′ by x'; then, $\quad O'A' = O''A' - O''O'$
or, $\quad A''A = OA - OA''$
or, $\quad x' = x - \alpha$

And as, by changing x into y or z the same reasoning applies to the other co-ordinates, it follows that we have only to make this change, in order to adapt the equation deduced for x' into equations that determine the values of y' and of z'; whence, denoting the co-ordinates of P, reckoned from the primitive origin, by x, y and z; the co-ordinates of P, reckoned from the new origin, by x', y' and z', and, finally, the co-ordinates of the new origin, reckoned from the primitive one, by α, β, and γ; we have,

$$\left.\begin{aligned} x' &= x - \alpha \\ y' &= y - \beta \\ z' &= z - \gamma \end{aligned}\right\} \quad \ldots\ldots 15$$

Sect. II. The elementary relations of points expressed in terms of their co-ordinates.

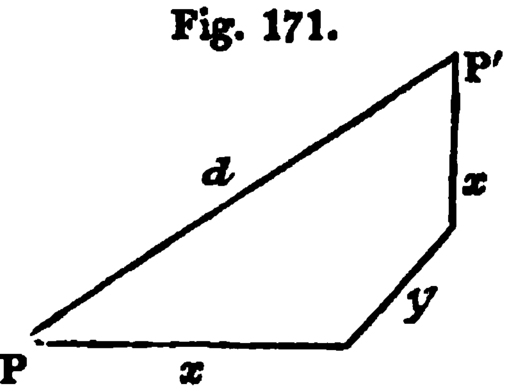
Art. 98. Theorem for shifting the origin.

This investigation completed, we may proceed to the problem suggested in art. 97.

99. *Two points being given by their rectangular co-ordinates, it is required to find the distance between them.*

If α, β, γ, are the co-ordinates of the first point, and α', β', γ', those of the second, it appears from the preceding problem, that, reckoning the first of these points as a new origin, the co-ordinates of the second will be,

Fig. 171.



$$x' = \alpha' - \alpha$$
$$y' = \beta' - \beta$$
$$z' = \gamma' - \gamma.$$

But the co-ordinates x', y', z', together with d, composing a closed figure, the formula 9, art. 74, by assuming the line m as coincident with d, will give,

$$d = x' \cos. x'd + y' \cos. y'd + z' \cos. z'd \quad . \; . \; . \; 16$$

and making the line m successively coincide with x, with y, and with z, and omitting the terms that contain angles whose cosines are zero, we obtain

$$\left.\begin{aligned} x' &= d \cos. dx' \\ y' &= d \cos. dy' \\ z' &= d \cos. dz' \end{aligned}\right\} \quad . \; . \; . \; . \; 17$$

Where, multiplying each of the equations by its left hand term, and afterwards subtracting the three last equations from the first, there arises,

$$d^2 = x'^2 + y'^2 + z'^2 \quad . \; . \; . \; . \; . \; 18$$

Or, putting for x', y' and z' their values, and taking the square root,

$$d = \sqrt{\{ (\alpha' - \alpha)^2 + (\beta' - \beta)^2 + (\gamma' - \gamma)^2 \}} \quad . \; . \; 19$$

We proceed to the second of the two leading problems announced at the commencement of this section.

100. *Required the angle formed by two given lines.* If the primordial elements are the same as in the preceding problems, and the lines denoted by r and r' are given in terms of the angles they form with the axes of the co-ordinates, we can solve the question by employing the co-ordinates of a point, taken at pleasure, in one of the lines.

For assuming a point in the line r, for example, denoting its co-ordinates by x, y and z; and making m coincident with r', the equation 9, art. 74, will give

$$r \cos. rr' = x \cos. r'x + y \cos. r'y + z \cos. r'z$$

and, by the equations 17,

$$x = r \cos. rx$$
$$y = r \cos. ry$$
$$z = r \cos. rz$$

Substituting and dividing off the common factor r,

$$\cos. rr' = \cos. rx \cos. r'x + \cos. ry \cos. r'y + \cos. rz \cos. r'z \quad . \; . \; . \; . \quad 20$$

When the lines are given in terms of the linear coordinates of points through which they pass, this formula will require to be modified, but the alteration is readily effected. For assuming the most general case, namely, that wherein two points are given in each line, and denoting the co ordinates of the points in the first line by x, y, z, x', y', z', and those of the two points in the second line by $x'', y'', z'', x''', y''', z'''$; we shall have, from the equations 15 and 17,

Sect. II. The elementary relations of points expressed in terms of their co-ordinates.

Art. 100. Angle formed by two given lines in terms of the elements that assign their direction.

$$\begin{aligned} x' - x &= r \cos. rx \\ y' - y &= r \cos. ry \quad . \; . \; . \; . \; 21 \\ z' - z &= r \cos. rz \\ x''' - x'' &= r' \cos. r'x \\ y''' - y'' &= r' \cos. r'y \\ z''' - z'' &= r' \cos. r'z \end{aligned}$$

Where r and r' are used, not only as the *names* of the given lines, but also to express the portions of them intercepted between the points in question.

Obtaining from the last equations the values of the cosines, and substituting them in 20, that expression becomes,

$$\cos. rr' = \frac{(x'-x)(x'''-x'')+(y'-y)(y'''-y'')+(z'-z)(z'''-z'')}{rr'}$$

and putting for r and r' their values, this result will finally reduce to

$$22 \; . \; . \; . \; \cos. rr' = \frac{(x'-x)(x'''-x'')+(y'-y)(y'''-y'')+(z'-z)(z'''-z'')}{\sqrt{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}}\sqrt{\{(x'''-x'')^2+(y'''-y'')^2+(z'''-z'')^2\}}}$$

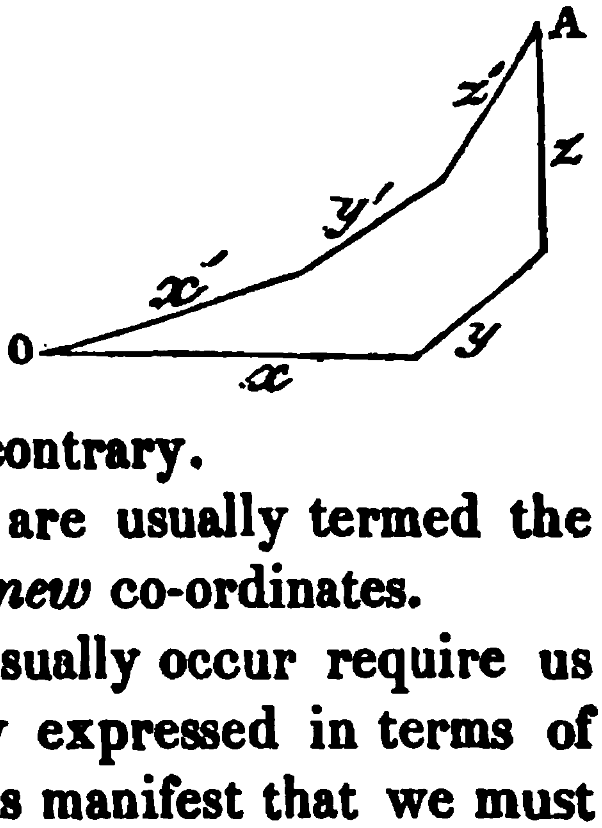
The expressions 19, 20 and 22 are the formulæ we proposed to investigate for the distance between two points, and the angles formed by two given lines: they admit of many transformations; but as their present arrangement is well adapted to the cases that most frequently occur, we shall not embarrass ourselves with the forms they assume when expressed in terms of co-ordinates of other kinds.

101. It will frequently happen, however, that when

the formulæ above given produce complicated results, a greater degree of simplicity may be obtained by using co-ordinates of the same kind, but having other directions; and as it is not always easy to determine, a priori, what directions of the co-ordinates will be the most advantageous, analysists have sought to express, in terms of a given system of co-ordinates, results that are given in terms of another system.

This operation is termed *transforming* the co-ordinates, and is of frequent use.

When applied to a system of linear co-ordinates, it is a method of expressing the position of a point by co-ordinates measured parallel to known lines, from having given its position in terms of co-ordinates measured parallel to other known lines.



Assuming A as the point to be determined; O as the origin, and x, y and z as the known co-ordinates, the question requires us either to determine new co-ordinates, x', y' and z', in terms of x, y and z, or the contrary.

The co-ordinates x, y and z are usually termed the *primitive*, and x', y' and z' the *new* co-ordinates.

And as the questions which usually occur require us to transform an equation already expressed in terms of their primitive co-ordinates, it is manifest that we must investigate, for this purpose, formulæ expressing the value of the primitive, in terms of the new co-ordinates.

The lines x, y, z, x', y', z', forming together a closed figure, the transformation will, in every instance, be readily effected, but as the case which supposes the original system to be rectangular is the only one of frequent occurrence and general utility, we shall confine ourselves

to that form of the problem; whence, by the theory of closed figures, we immediately obtain,

$$x = x' \cos. xx' + y \cos. xy' + z' \cos. xz'$$
$$y = x' \cos. yx' + y' \cos. yy' + z' \cos. yz'$$
$$z = x' \cos. zx' + y' \cos. zy' + z' \cos. zz'.$$

If the origin is changed at the same time as the directions of the axes, we have merely, by art. 98, to add to these values of x, y and z, the corresponding co-ordinate, α, β or γ of the new origin.

When all the points sought lie in a single plane, the latter may be chosen as one of the co-ordinate planes, the plane of the xy's, for example, in which case z will be zero. The formulæ then become,

$$x = x' \cos. xx' + y' \cos. xy'$$
$$y = x' \cos. yx' + y' \cos. yy';$$

and if both the new and the primitive system are rectangular, these expressions reduce to

$$x = x' \cos. xx' - y' \sin. xx'$$
$$y = x' \sin. xx' + y \cos. xx'.$$

PRELIMINARY REFLECTIONS TO SECTION III.

Applying the formulæ of the last or preceding sections to the relations of a given number of points, the problem may always be reduced to equations; but varying the positions of the points, not only do different cases of the problem arise, but the equations undergo corresponding variations. Are we then to write the equations of each case, or would it not be possible to obtain a rule for deducing the equations of one case from those of another? And if this is possible, might not the application of that rule change any deductions from the equations of one case into deductions applicable to another?

INQUIRIES SUGGESTED BY THESE REFLECTIONS.

Assuming one of the most general cases of a problem as a type of them all—deducing the equations of the type, and thence obtaining algebraic expressions for the properties sought: it is required, first, to trace the alterations which the type undergoes whilst passing into the other cases of the problem: secondly, to trace the corresponding alterations in the equations of the type: thirdly, the alterations in the algebraic expressions of the properties sought: and, lastly, to discover a rule whereby either of the two latter species of variations may be found from the former.

SECTION III.

THEORY OF CORRELATIONS.

Problems are resolved by a particular case of the problem taken as a type—the analysis of a geometrical proposition resolves itself into two parts; first, the analysis of a type peculiar to the proposition, secondly, an inquiry into the changes which the type undergoes—the first branch of this double analysis performed by regarding the type as composed of one or more closed figures—method of auxiliary elements—the analysis performed by primordial elements—the second branch of the analysis, or the method of correlations—demonstration of a general rule which connects the changes of the diagram with those of the equations—correlations of angles—angles greater than unity—sequence in which lines and angles are to be estimated—various forms of the theorem which expresses the relations of closed figures—correlation of figures that are used simultaneously—method of avoiding the superfluous equations that would result from the preceding rules.

102. We explained in art. 51, and in the several articles of section 5, chapter 1, in what manner the most complicated figures could be compared with the same ele-

Chap. II. Of the elements to which place is referred.
Art. 102. Problems are resolved by a particular case taken as a type.
Art. 103. Every proposition resolves itself into, 1st, the analysis of a type peculiar to the proposition; 2d, an inquiry into the changes of that type.

mentary type or standard; and we have now only to show that, together with this general standard, common to all the figures of geometry, each problem has a type peculiar to itself, or, to speak more correctly, we must show that geometrical reasonings are performed by assuming such a type.

103. The analysis of triangles, as partially developed in the third section of the first chapter, will assist us in demonstrating this assertion.

It is there observed that a triangle, according to the proportions of its sides, admits an infinite variety of forms, passing into each other by insensible gradations.

And we conclude from this fact that a proposition relating to triangles will require two distinct processes for its solution—namely,

An investigation performed on some particular triangle:

And, an inquiry into the modifications which this investigation requires, when the triangle is made to pass through all the transformations of which triangles admit.

Similar conclusions will be true for other figures; and we may regard the analysis of all geometrical propositions as consisting of—an investigation performed on a particular case chosen as a type—and, an inquiry into the connections of the type with the several cases of the problem.

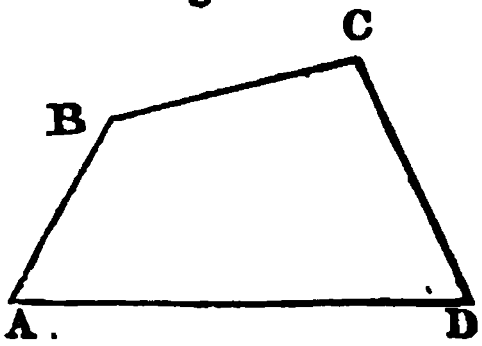
104. The former of these investigations, we have already remarked, is conducted by resolving the problem into the relations of one or more closed figures; and we have provided, art. 74, an instrument by whose

assistance the properties of such figures can instantly be assigned in equations.

To illustrate what is here said by an example, let us suppose the problem requires us to deduce certain relations of four points from other relations of three points that are given.

Connecting the points by straight lines drawn according to the order explained in art. 40, we form a closed polygon, the number of whose sides agrees with the number of given points, and applying to this figure the formula of art. 74, and making the arbitrary line m identical with one of the sides, we obtain an equation that involves certain relations of the figure.

Fig. 173.



Thus, if m is made identical with a, the resulting equation is

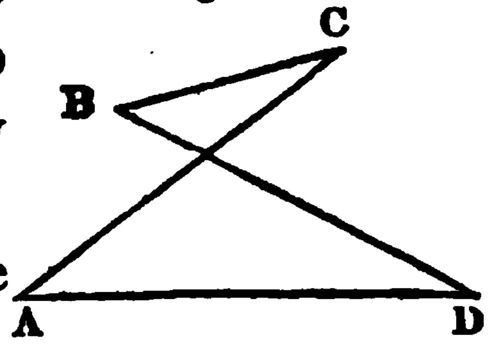
$$a = b \cos.\ ab + c \cos.\ ac + d \cos.\ ad.$$

And as we may cause m to coincide with any other side of the figure, we shall, on the whole, obtain four equations, whence, with sufficient data, the relations sought may be determined.

But the polygon obtained by joining in sequence the four points ABCD, is not, as we have remarked in art. 40, the only closed figure found among the relations of those points.

Others may be formed by connecting them in a different order, and from the figures so constructed, by a process similar to that used above, similar equations may be deduced.

Fig. 174.



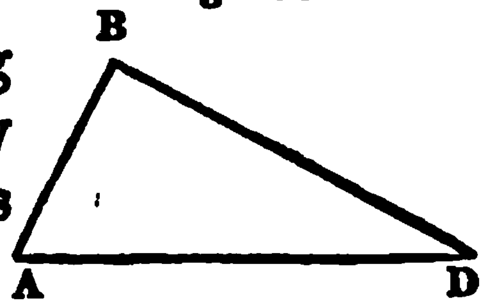
Nor have we yet exhausted the

equations to which an analysis of the problem before us gives rise.

Other equations can be deduced by the use of other methods; by forming closed figures that do not embrace *all* the points, and by employing elements that are afterwards eliminated.

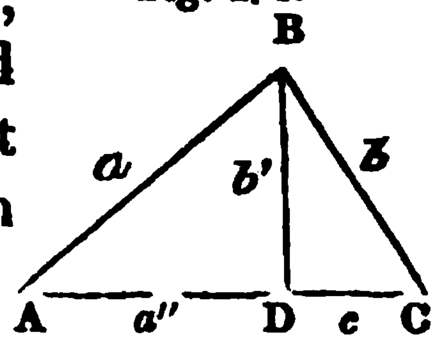
Thus ABD is a closed figure formed out of the relations of three among the four points A, B, C, D: and by varying the relation other triangles may be obtained.

Fig. 175.



105. The method of auxiliary elements alluded to in the preceding article, consists in forming additional closed figures, by drawing lines that are not found among the relations of the given points, and by eliminating at a subsequent period of the investigation the angles and lines such additional elements have introduced. A simple example of this method occurs in the analysis of triangles; where, letting fall a perpendicular BD, we analyse the triangle into two closed figures ABD and BDC: from the first of which the formula of the forty-ninth article deduces the equation

Fig. 176.



$$b' = a \cos. ab' = a \sin. aa''$$

and from the second figure, by the same article, there results,

$$b' = b \cos. bb' = b \sin. bc;$$

whence, eliminating b', and observing that AD, or a'', agrees in direction with AC, or a', whilst CD, or c, has the contrary direction, we obtain

$$\frac{a}{b} = \frac{\sin. ba'}{\sin. aa'}.$$

Similar equations expressing the ratio of any two sides may be obtained by the same process, but the method of analysis we have been considering will be fully comprehended from the preceding remarks, and we may dismiss the subject with an observation respecting the selection of the type or standard diagram: a selection in this respect becomes necessary, from the fact that in particular positions of the points, relations of them may disappear; and it is important that we should find in the type all the parts that belong to the most general case of the problem.

The selection of the standard diagram must be made with a view to this object, which accomplished, the choice of the figure is in all other respects arbitrary.

In this analysis no mention has been made of primordial elements, but we are not on that account to suppose the method independent of them.

All methods of geometrical analysis proceed on the supposition of primordial elements.

But in some cases the lines proper to the figure are used as standards of position; in others the elements of reference are either arbitrarily introduced, or, if governed in their position by the nature of the problem, are otherwise foreign to it.

An analysis of the latter kind is chiefly employed when the given points are numerous; when general reasonings, rather than particular results, are required; and, finally, when the problem falls in the *indeterminate analysis*, an extensive branch of our subject that will occupy us in another chapter.

We have thus two distinct methods of analysis; in one of which the primordial elements do not directly appear,

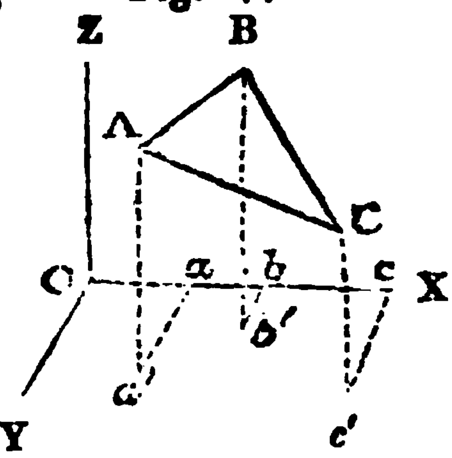
whilst in the other every step of the investigation is conducted with reference to them.

We have explained the application of the first of these modes of inquiry to the problem enunciated in art. 103: and it now remains to explain the application of the second method to the same problem.

106. For this purpose let us investigate the relations of three points, employing three rectangular axes for primordial elements.

According to the rules that are laid down on this subject in art. 84, A is assigned by means of its co-ordinates x, y and z; or Oa, aa' and a'A; B by similar co-ordinates x', y' and z'; and C by the co-ordinates x'', y'' and z''.

Fig. 177.



And so convenient is the method we are describing, as an artifice for expressing position and quantity by words and letters, that already, at this early stage of our explanation, the reader must have perceived its peculiar adaptation to that purpose. Let us suppose, for example, any number of points, denoted, according to the principles established in page 109, by A, B, C, D, &c.: we can express the position of any *one* of these points by three co-ordinates x, y and z; and placing accents over these characters, and making the *number of accents* agree with the *rank* of the letter denoting the point, we can assign, without fear of error, the point to which the co-ordinates belong. The whole arrangement is more fully expressed in the following table:

Rank of the letters,	0	1	2	3 &c.
Denominations of the points whose relations are given or sought,	A	B	C	D
Co-ordinates measured on the axe of the x's,	x	x'	x''	x'''
Co-ordinates measured on the axe of the y's,	y	y'	y''	y'''
Co-ordinates measured on the axe of the z's,	z	z'	z''	z'''

As, then, any problem in the determinate analysis, however complicated, has been proved to be merely the relations of points, we can always express the data of such problems by a table.

When the given things are the positions of the points themselves, the table will merely contain the values of x, of y, of z; of x', of y', of z', &c.; and from such data the formulæ investigated in the beginning of this section will immediately deduce the relations of the problem.

Thus, returning to the relations of three points, let us suppose the co-ordinates of A, B and C to be known; and, as known quantities are generally represented by first letters of the alphabet, let our table be,

$$\begin{array}{lll} x=\alpha & y=\beta & z=\gamma \\ x'=\alpha' & y'=\beta' & z'=\gamma' \\ x''=\alpha'' & y''=\beta'' & z''=\gamma'' \end{array}$$

The sides of the triangle will be found by the formula 19, which gives,

$$a=\sqrt{\{(\alpha'-\alpha)^2+(\beta'-\beta)^2+(\gamma'-\gamma)^2\}}$$
$$b=\sqrt{\{(\alpha''-\alpha')^2+(\beta''-\beta')^2+(\gamma''-\gamma')^2\}}$$
$$a'=\sqrt{\{(\alpha''-\alpha)^2+(\beta''-\beta)^2+(\gamma''-\gamma)^2\}}$$

and the remaining relations of the points, the three an-

gles, can be found from the formula 22, by the same substitutions.

This method is, at once, so simple and general, that any further illustration of it will not be required; nor will greater difficulty be experienced when, instead of the position of the points being given, our data consist of their relations: let us suppose, for example, that among the relations of A, B and C, the distances a, b and a' were given, the equations

$$\begin{aligned} a &= \sqrt{\{(\alpha' - \alpha)^2 + (\beta' - \beta)^2 + (\gamma' - \gamma)^2\}} \\ b &= \sqrt{\{(\alpha'' - \alpha')^2 + (\beta'' - \beta')^2 + (\gamma'' - \gamma')^2\}} \\ a' &= \sqrt{\{(\alpha'' - \alpha)^2 + (\beta'' - \beta)^2 + (\gamma'' - \gamma)^2\}} \end{aligned}$$

would suffice to determine α, β, γ, &c., and from these we could find, by the last example, the angles of the triangle.

And, generally, whatever are the number of points, their relations will consist of lines and angles; and such of these relations as are given can be expressed by means of the formulæ 19 and 22.

These formulæ are so many equations, and thence, if the data are sufficient, we shall be enabled to determine the co-ordinates of the points.

Which done, the preceding problems offer us the means of discovering the remaining parts of the problem.

The analysis we have explained reduces to fixed and known rules the solutions of all determinate problems; and nothing remains of an arbitrary nature in the process, except the position of the origin, and the directions of the co-ordinates; nor with respect to these subjects are rules altogether wanting; although in the application of such rules we must be guided in a great measure

by the nature of the problem, and by considerations that can be more advantageously discussed in another place.

107. The investigation of the type, or standard figure completed, we pass to the second branch of the analysis of problems, announced in art. 103, and which, as requiring us to trace the relations whereby different cases of the same problem are connected, has been termed the method of *correlations*.

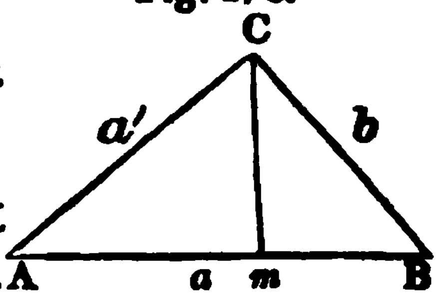
This branch, it will be recollected, is immediately connected with that we have just considered; and has for its object to render the application of the latter general, by considering the changes incident to the standard diagram, when this last is made to represent, successively, all the cases of the problem.

Some idea of these changes, and the artifice used to compensate them, may be obtained from the partial investigation of triangles that occupied us in art. 51.

From the triangle ABC we there deduced the formula

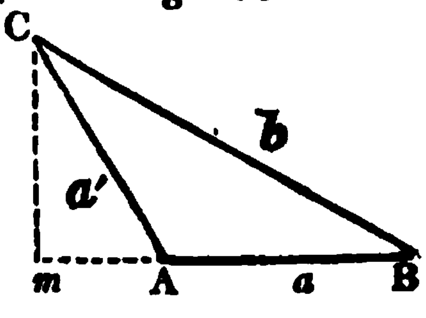
$$a = a' \cos. aa' + b \cos. ab.$$

Fig. 178.



And considering this triangle as a type, and comparing it with the triangle 179, where the point m falls without the angle A, we observed that a difficulty occurred respecting the inclination of the lines a and a'; a difficulty removed in the article in question by estimating this inclination by the angle BAC.

Fig. 179.



The motive assigned for this choice applies immediately to the subject before us.

"The true reason," it is observed, "for choosing this measurement is the power it gives of extending to the

present case the formulæ deduced from the preceding."

And, in support of this assertion, and of our right to make the choice alluded to, it is further remarked,

That, as, "a right angled triangle cannot contain an obtuse angle, (it follows) art. 50, that such angles will neither have sign nor cosine unless a further convention is made respecting those terms."

The additional convention here alluded to, was altogether arbitrary, since it could not interfere with the definition already given, which applied to acute angles only.

But agreeing that an angle has the same cosine as its supplement, but with a contrary algebraic sign perfixed to it, we deduce, as in the art. referred to,

$$Bm = b \cos. ba$$
$$Am = -a' \cos. aa'$$

And by subtracting these equations there still results the same equation as in the former case, namely

$$a = a' \cos. aa' + b \cos. ab$$

By this artifice, of agreeing to write a negative sign before the cosine of an obtuse angle, one formula has been extended to two distinct cases; and examining further we shall find, that, not only two, but all cases of triangles, are brought by this simple agreement to admit the formula

$$a' = a \cos. aa' + b \cos. ba'$$

It is not at present an easy task for the student to form a just notion of the important results obtained by condensing in this manner the various cases of a problem.

When further advanced many illustrations of it will occur.

An elementary theorem, that will occur in Part III. for example, admits thirty-two cases—a similar number belongs to a theorem connected with it, and a proposition where both these theorems were combined would admit thirty-two times thirty-two, or, not less than, one thousand and twenty-four distinct cases.

This last example would alone demonstrate the value of the artifice we have employed; and, prove that investigations may thus be condensed into a page that otherwise would have occupied volumes.

But an instrument so important ought not to be dismissed until we have given it the most advantageous form of which it is susceptible.

On this account we will consider it in another point of view.

Returning to the proposition which gave rise to this digression, and comparing the cases designated in figures 108 and 109; we observe that Am in these two figures falls on the opposite sides of A; and as

$$\frac{\mathrm{A}m}{\mathrm{AC}} = \cos.\ aa'$$

the artifice of the negative cosine would be satisfied by agreeing to reckon Am positive when it fell upon a certain side of A, and negative when it fell upon the other side.

Viewed in this light the agreement in question would lead to the rule, that—

A change of direction requires a change of sign.

This rule we shall examine more closely; commencing our investigation with the analysis by primordial elements.

Some observations respecting the latter which have

occurred in the preceding pages, will assist in elucidating the discussion.

We have there remarked that all geometrical problems refer to the relations of points.

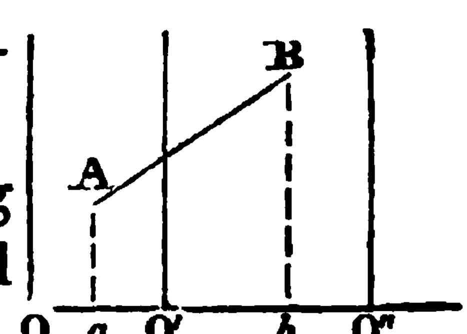
And, whilst treating of the analysis by primordial elements, we observed, art. 106, that by a due selection of those elements, the relations of points could be expressed in terms of the differences of their co-ordinates.

The chief selection regarded the position of the origin.

And our first inquiry concerning the changes of the figure ought therefore to have in view the effect produced by an alteration of the origin.

Suppose A and B the points in question—the origin at O, then $Oa = x$, $Ob = x'$, and ab the difference of the ordinates is expressed by $x' - x$.

Fig. 180.



But shifting the origin, and taking it at O′ we have $O'a = x$, $O'b = x'$; and ab, no longer a difference but a sum, is expressed by $x' + x$.

When the origin is at O″ we have

$$O''a = x$$
$$O''b = x'$$

and ab, which again becomes a difference, is expressed by

$$ab = x - x'.$$

These three values of ab may be regarded as cases of the same problem, and comparing the changes in the equations with those in the figures, we can show that our rule causes the former variations to depend upon the latter.

The change in the equations $ab = x' - x$ and $ab = x' + x$ consists of an alteration in the sign of x: the variation in the diagrams, at least that part of it which renders the same equation no longer applicable, is produced by an alteration in the direction of x, but the rule, consistent with this result, in deducing the change of the equation from the change of the figure, alters from plus to minus the algebraic sign of those lines that reverse their direction.

And, finally, comparing, in like manner, the third equation with that deduced from the type, we observe, not only, that we obtain $ab = x - x'$ from $ab = x' - x$ by changing the signs of x and x', but also, that in one diagram x and x' are measured in directions the reverse of those they occupy in the other. And this case, therefore, as well as the former, is consistent with the rule.

108. These partial examples, whilst they serve to make the reader acquainted with the artifice that has occupied our attention, ought not to be regarded as a demonstration of it.

A demonstration, however, may be gathered from the preceding pages.

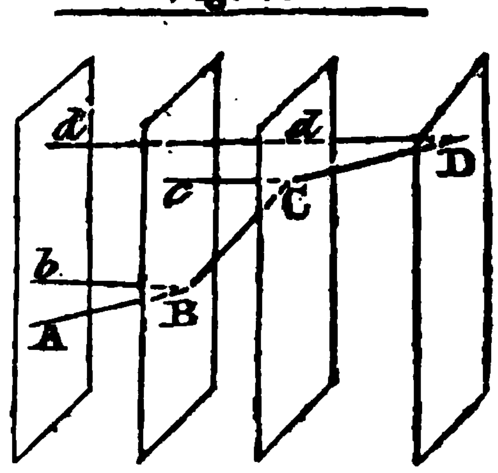
All geometrical investigations, it has been then shown, are dependent on closed figures, and, ultimately, on the theorem of art. 74.

Turning, then, our attention again to this theorem, if we can prove that all the cases of it are embraced by the artifice in question, the latter may be regarded as applicable to every geometrical investigation.

Chap. II. Of the elements to which place is referred.
Art. 108. Demonstration of a general rule which connects the changes of the diagram with those of the equations.

But this has, in fact, been accomplished in establishing the theorem itself, as will appear by attending to what is there said concerning the line DC.

Fig. 181.



As, however, the subject is important, it may not be amiss to repeat the reasoning used in art. 74, and to add to it some further illustration.

The points A, B, C and D, falling as in figure 181, we deduce from the construction used in art. 74

$$(\alpha) \quad . \; . \; . \; . \quad Dd' = Bb + Cc + Dd.$$

But the position of the points whose relations we propose to investigate is often unknown : and if in solving a given problem concerning the relations of four points, we assume them to lie in the order adopted in figure 181, that assumption may be erroneous; and thus, we are uncertain, in such cases, how far the equation α is to be depended on.

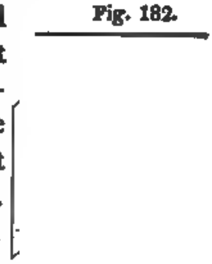
The method we are seeking to establish proposes to solve this difficulty.

It teaches us to regard figure 181 as a type with which all cases of the problem should be compared.

And directing us to deduce from this type the equation α, asserts, first, that equation to be applicable in all cases where the perpendiculars, Bb, Cc, &c. have the same directions as in 181; and, secondly, that to render the equation true, for those cases where Bb, Cc, &c. change their directions, it is only necessary to regard as negative those perpendiculars whose directions are reversed; or, art. 74, to measure the inclination of AB,

BC, &c. with m, by the larger of the two adjacent angles.

Fig. 182.



A comparison of figures 181 and 182, will establish the truth of what is here said, not only for the particular number and arrangement of the points delineated in those figures, but for any number and any arrangement.

For, although on comparing 181 and 182 we find but one line, Cc, that changes its direction, and but one term of the equation, (Cc), that becomes negative; yet, if in this simple case a change of direction produces a change of sign, it will do so in all other cases, since the most complicated examples can only present many lines that change their directions, and corresponding terms that become negative.

The artifice in question *is*, therefore, generally applicable to the theorem of art. 74; it *does*, whilst the number of points remains the same, enable us to combine all the cases of that theorem in one case; and we must conclude, in the words of a preceding paragraph, that the artifice is applicable to every geometrical investigation.

Or, as we may more exactly word the result,

An equation derived from the most general case of a proposition applies to every other case, provided, on comparing the diagram of the former with the diagram of the latter, all lines in this last are reckoned negative, the projections of which, on a given straight line, appear to have changed their directions.

But without some further remark on this rule, we shall not perceive its application to the case, with a view to which it was constructed.

Chap. II. Of the elements to which place is referred.

Art. 108. Demonstration of a general rule which connects the changes of the diagram with those of the equations.

In the instances alluded to, we were supposed ignorant concerning the arrangement of the points, whose relations were the object of investigation.

But if the arrangement of the points is unknown, the directions of the perpendiculars let fall from them must be unknown, and the rule seems inapplicable.

A theorem demonstrated in the elements of algebra will remove this difficulty.*

Particular errors in the statement of a numerical proposition, it is there shown, will not affect the quantity of the results, but merely affix signs to them, indicating the corrections required in the premises. Now these errors are precisely such as occur in the problem before us; and arise when the quantities sought are considered additive, instead of subtractive,—the error in the premises being then denoted by a negative sign attached to the answer.

The rule we have devised may, therefore, be neglected until our investigations are complete, and will thus enable us to interpret the results, and compare the figure analysed with the type assumed to represent it.

Another, and final observation respecting this rule, is introduced to warn the reader against an error that has deceived a mathematician of eminence.

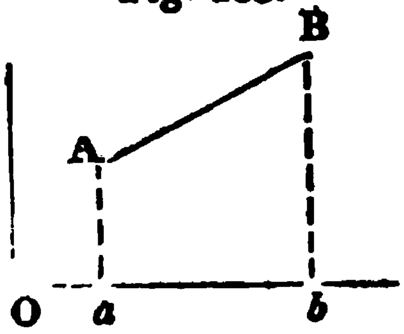
Carnot has supposed geometricians to regard certain lines as "essentially negative," an error not to have been expected in a writer who usually displays so much perspicuity.

Positive and negative are other words for additive and subtractive; operations to which all lines are equally adapted: and in place of regarding one direction as po-

* Note 5.

sitive, and another as negative, a more just view of our subject will, I trust, teach "us" to regard *a change of direction as compensated by a change of sign.*

Fig. 183.



A simple example, already employed, will illustrate what is here said. In investigating the problem discussed in art. 107, we selected as a type the case corresponding to fig. 183 and thence deduced the equation

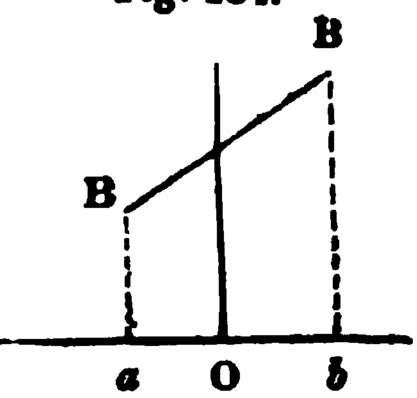
$$ab = x' - x.$$

But fig. 184 might with equal propriety have been chosen as a type; the equation would then have been,

$$ab = x' + x$$

and in establishing it we should have added together lines whose directions are opposite; we should have added Oa and Ob, without supposing the difference in the directions of these lines, to render one positive and the other negative.

Fig. 184.



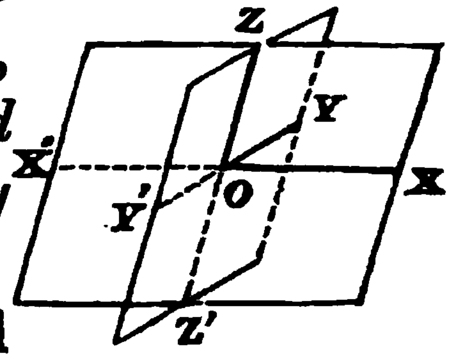
But, comparing the first diagram with the second, regarded as a standard, we observe that Ob or x' has the same direction in each, whilst Oa changes its direction, being opposed, in the figure chosen as a type, to the direction of ab, and agreeing with that direction in fig. 183: this change of direction alters the expression for ab, and must be compensated by a change of sign; Oa, regarded as positive in figure 184, must be considered as negative in figure 183; and, with this convention, the equation $ab = x' + x$ will equally belong to either case.

The illustration by rectangular co-ordinates of the artifice discussed in this and the preceding article, has

hitherto been confined to a single axe, but what has been said concerning one axe applies to either of the three, and is not affected by the angles at which they are inclined.

Hence, in any system of axes, whether rectangular or oblique, the following rules may be observed.

Fig. 185.



The origin, O, may be so chosen that every point belonging to the type, or figure of comparison, shall be found in one of the eight angles formed by the co-ordinate planes (art. 67).

Let the solid angle be that formed by ZOX, ZOY, YOX.

In comparing any other case of the problem with this type, the ordinates x are to be reckoned positive when measured towards X, and negative towards X';—the ordinates y are positive on the side of Y, and negative on that of Y';—finally, the ordinates z are positive on the side of Z, and negative when measured towards Z'.

The general method of analysis proposed at the end of the 103d art. having been examined and established in the preceding pages, we are in possession of an instrument by which geometrical propositions can be instantly reduced to equations.

This analysis, we recollect, consisted of two branches, and in establishing the second branch, or the method of correlations, we first proved the artifice of negative cosines to be identical with that which regards a change of sign as indicating a change of direction; and afterwards, adhering to this view, demonstrated the justice of applying to every case of a proposition the equations deduced from that particular case which served as a type.

Sect. III. Theory of correlations.

Art. 108. Demonstration of a general rule which connects the changes of the diagram with those of the equations.

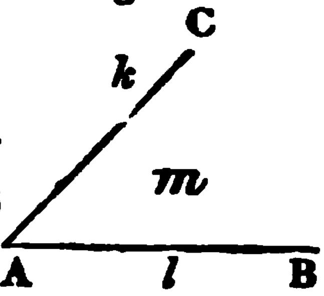
But, in treating this subject, the change produced among the angles of a figure by an alteration in the arrangement of the given points, has been considered less fully than it deserves.

Angles, no less than lines, are additive, and admit a change of direction; and we should inquire whether the laws regulating the correlation of lines do not apply to angles also.

The subject was touched upon in what was said concerning negative cosines, but we will investigate it at more length.

109. An angle, we have seen, art. 42, is measured by the portion of plane space included between its sides when infinitely produced.

Fig. 186.

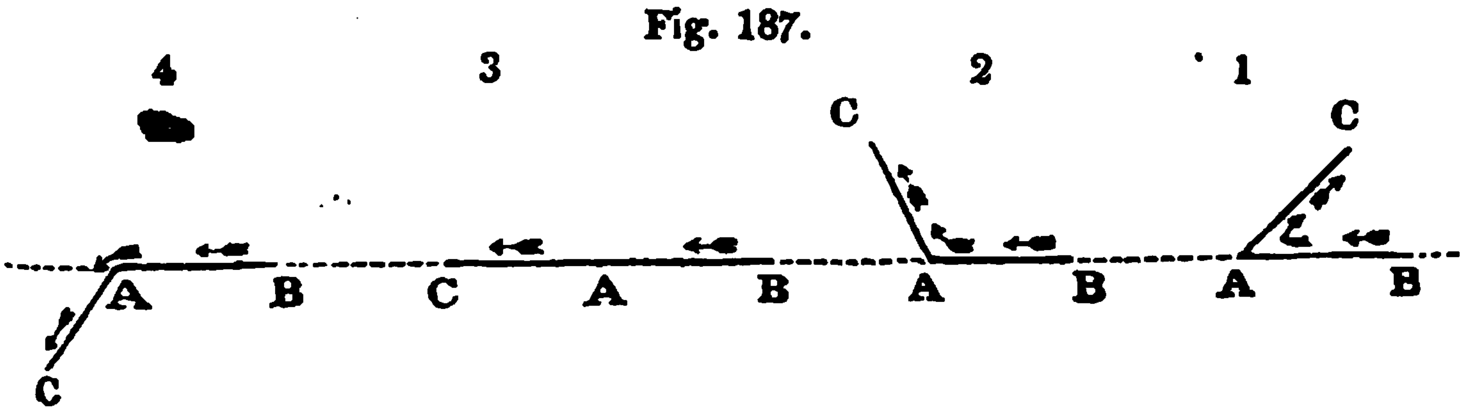


But the plane space about A is divided into two unequal parts by the directions AB and AC: the least of these parts is the space bounded by AB and AC, but indefinite towards *m*; and the larger part is the space bounded also by AB and AC, but indefinite in the directions *k* and *l*.

The ratio of either part to the whole of plane space, may be taken for the inclination of AB and AC; nor is the selection important.

Let us choose the smaller of these ratios; and, supposing AB to revolve about A, examine its successive values.

Fig. 187.



In figs. 187—1 and 2, we observe the area whose ratio to plane space measures the inclination of AB and AC successively increasing, until, in fig. 187—3, AB and AC form one right line; the space about A is thus divided into two equal parts: but in the next figure, 187—4, the space CAB, of fig. 187—1 has increased until it has become the *greater* of those two parts into which the lines BA, AC divide the plane space about A.

The figs. 187—1, 2, 3 and 4 are cases of the same diagram, corresponding to different positions of C.

And thus, in tracing the successive positions of that point, either, we must distinguish two cases, according as C falls on one or the other side of AB, or admit that an angle, as in fig. 187—4, may be measured by the greater of those spaces into which the plane space about its vertex is divided.

Either convention is consistent with the measure of an angle given in art. 42, but other considerations will limit our choice, and the second convention must be adopted, as alone in accordance with the method which deduces every case of a proposition from a standard diagram or type.

An angle, then, may exceed unity; and this result, we again repeat, is consistent with art. 42, where the whole of plane space was shown to be the natural unit of angles, and any angle less than the unit was expressed *geometrically* by the opening between the directions that formed it, and *analytically* by the ratio between this opening and the whole of plane space.

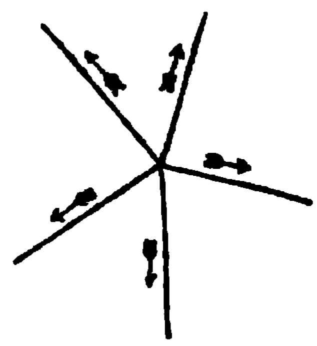
110. Expressed by this ratio, we have shown that an angle may increase from 0 to 1, but even when it has attained that value, we may suppose other angles, art. 41,

added to it, and thus obtain an angle greater than unity. Or, to view the subject as illustrated by the 187th and following diagrams, we may suppose the line AB, after revolving, as above described, through the whole of plane space, to regain its first position and commence the course anew.

But an angle greater than unity is not readily expressed by a diagram, and although the idea of such an angle has been seen to follow from those of endless rotation and continued addition. This idea ought yet to be regarded in the light of an analytical artifice.

When many lines diverge from a common point, the rules here given will direct us to measure their inclination by the angles included between the directions themselves, and not to confound, by producing either of the lines, a direction with its converse. With this agreement, and provided the direction of each line is completely assigned, no ambiguity can occur with regard to their mutual inclinations.

Fig. 188.



The most exact method of assigning their directions is that of primordial elements; which may be employed under any of the forms explained in art. 100.

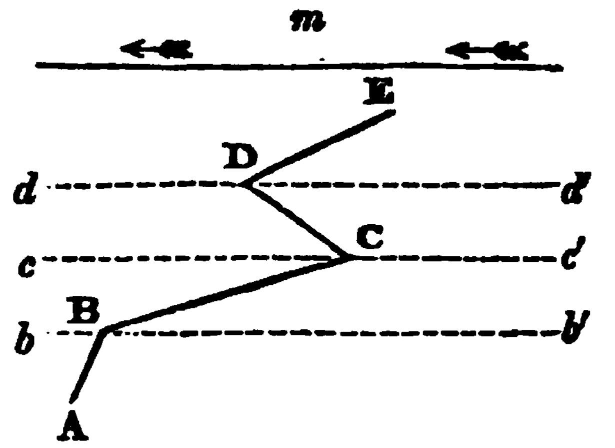
When the positions of the lines are not explicitly given, but are consequences of the mutual inclinations assigned to them, the precision or ambiguity of the results will depend on the data, and the hypothesis may even be consistent with more than one position of the same line.

111. The minute discussion we have given to this subject, will have prepared us to appreciate more fully a principle employed in the 74th art., and introduced there with the express intention of removing all ambiguity respecting the measurement of angles.

Chap. II. Of the elements to which place is referred.
Art. 111. Sequence in which lines and angles are to be estimated.

This principle consists, first, in estimating according to a fixed sequence, all the points whose relations are the objects of inquiry; and, secondly, in substituting direction, for the more vague term, right line.

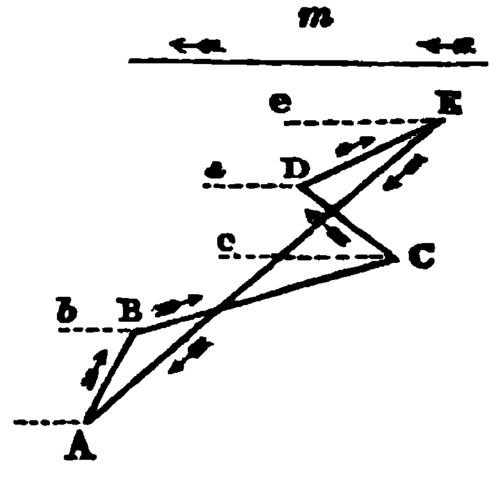
Fig. 189.



Thus, denoting the direction of m by the arrows annexed, and estimating that of AB, of BC, &c. by the sequence of the points; we are no longer embarrassed by the ambiguity otherwise attending the distinction between an angle and its supplement, but can immediately decide between ABb, for example, and ABb', or CDd and CDd'; since Bb' and Dd' are to be rejected as contrary to the direction of m.

Applying this principle to the theorem demonstrated in art. 74, we immediately perceive that one of the sides of the closed figure, the last side, is estimated according to a direction opposed to that of the other sides, or, contrary to the sequence of the points.

Fig. 190.



Closing, for example, by means of the line EA, the open polygon 189, the rule at which we have arrived supposes us to proceed as follows:

1. The sides are estimated in the sequence of the points, or in other words, according to the order and directions AB, BC, CD, DE, EA.

2. Calling that extremity of a line at which we arrive last, or which is denoted by the superior letter, page 109, the "superior extremity of the line," we draw through the superior extremity of each side a line parallel to, and directed the same way with m.

3. The inclination of any side to m is then estimated by taking in their due directions the side, and the parallel which passes through its superior extremity. The inclination, for example, of a to m, fig. 190, is measured by the angle which AB and Bb include between them.

Applying this rule to each of the sides that are found in fig. 190, we observe, as above noticed, the inclination of the final side EA to be measured by an angle EAa, supplementary to that, AEa, which is assumed as the measure of this inclination in art. 74.

The rule however which we have here explained, possesses so much simplicity, that we shall adopt it as our future guide, and reconcile to this method the theorem of art. 74 by regarding the final side in the latter as having an *inverse direction.*

112. With these remarks we might terminate our preparatory operations, and proceed to employ the instruments in constructing which we have hitherto been engaged. But the universal application ascribed to the theorem of art. 74 is a sufficient excuse for developing, in the most ample manner, both the theorem itself, and the method of employing it; and, accordingly, I shall terminate what is here said with an enumeration of the most useful forms under which the theorem in question occurs; and afterwards, in a separate article, proceed to examine the cases of correlations that arise when more than one closed figure is used in the same analysis.

The theorem of art. 74, whose most useful cases we propose to exhibit, may be expressed as the following property of closed figures.

I. *In any rectilinear closed figure, if each side is multiplied into the cosine of the angle it forms with a com-*

mon but arbitrary direction, either of these products will be equal to the sum of all the products remaining.

The notation explained in art. 39, enables us to express this rule under a form more convenient for practice.

RULE FOR WRITING ALGEBRAICALLY THE THEOREM 9, FORM 1.

First. *Write as an incomplete member of an equation the symbol denoting any side of a closed figure; and commence the other member by writing there, at intervals, the remaining sides.*

$$a \quad = \quad b \quad c \quad d \quad \&c.$$

Secondly. *Expressing, by a convenient symbol, any fixed but arbitrary direction; multiply each letter of the equation by the cosine of an angle formed of that letter and the symbol last mentioned,*

$$a \cos. am \quad = \quad b \cos. bm \quad c \cos. cm \quad d \cos. dm \quad \&c.$$

And lastly. *Connect by the sign of addition the several terms on the right hand side,*

$$a \cos. am = b \cos. bm + c \cos. cm + d \cos. dm + \&c.$$

II. Assuming the arbitrary line m to coincide with one of the sides of the closed figure, a, for example, when the angle between them will of course vanish, we shall obtain another form of the theorem, and one very generally employed.

THEOREM 9, FORM 2.

$$a = b \cos. ba + c \cos. ca + d \cos. da + \&c.$$

The rule for obtaining this result may be deduced from the former rule by removing the second section, and substituting in its place the following words:

Secondly. *Multiply each letter of the equation by the cosine of an angle formed of that letter and the letter in the left hand member.*

III. A third, and in some respects the most remarkable form of this important instrument may be obtained by transposing the term in the left hand member, and reversing the direction according to which the side, so transposed, is estimated: by this change of direction we alter the sign affecting the term in question, and the equation becomes,

$$0 = a \cos. am + b \cos. bm + c \cos. cm + \&c.$$

In this form of theorem 9, as we shall perceive by referring to what is said in art. 111, the final side is reckoned in the same sequence as the other sides.

IV. By varying the position of m these theorems receive various modifications.

Assuming m at right angles to a plane which passes through a, the cosines of the angle made by the sides and this perpendicular are changed into the sines of the angles which the sides make with the plane, and there results,

$$0 = b \sin. \mathrm{M} + c \sin. \mathrm{M} + d \sin. \mathrm{M} + \&c.$$

where M is the given plane.

113. Making the arbitrary line m agree successively with each side of a closed figure, the relations of which are sought, we obtain from the second form of theorem 9 as many equations as there are sides in the figure: and since all the parts of the latter are involved in the equations, these last, when a sufficient number of the parts are given, will enable us to find the remainder.

The polygon ABCD, fig. 173, page 229, for example, analyzed in this way produces the equations

$$a = b \cos. ba + c \cos. ca + a'' \cos. a''a$$
$$b = c \cos. cb + a'' \cos. a''b + a \cos. ab \ldots (m)$$
$$c = a'' \cos. a''c + a \cos. ac + b \cos. bc$$
$$a'' = a \cos. aa'' + b \cos. ba'' + c \cos. ca''$$

And since these equations contain all the sides and angles of the polygon, the values of the unknown parts will be found, by elimination, in terms of parts that are given.

So far the process seems direct and clear; but as the relations of the points A, B, C, D, may, art. 40, be analyzed into a second closed figure, 174; a new set of equations can be deduced.

And the question for us now to determine is, whether the equations obtained from the figure 173 are simultaneous with the equations obtained from 174.

In both sets of equations the same lines and angles occur, but we may doubt whether they are estimated in the same manner.

That is to say, we might doubt whether the rules laid down in art. 111, for distinguishing between an angle and its supplement, and for determining the directions of lines, would give, when applied to figure 174, the results they gave for the case exhibited in 173.

To examine this doubt with the attention it deserves, we must recur to the method of analysis, proposed, at the end of art. 103, and, established, in the pages immediately following that article.

The analysis developed there requires us to draw all our conclusions from one type; and by constantly keeping this principle in view we shall not fail to remove the difficulties attendant on particular cases.

Sect. III. Theory of correlations.

Art. 113. Correlation of figures that are used simultaneously.

In obtaining the equations m, for example, the truth of our results might have been demonstrated by the comparison of figures, directed in the article alluded to.

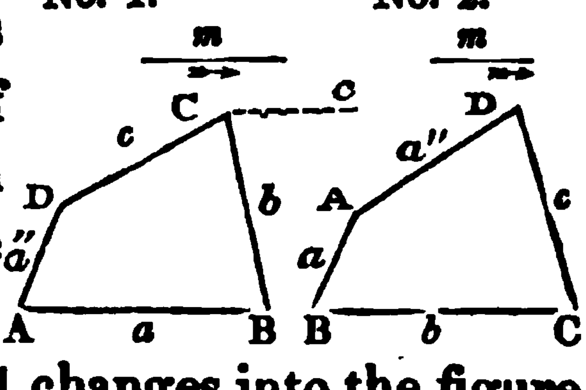
The method we actually pursued, it will be recollected, was to deduce, by a transposition of letters, the three last equations from the first: and we supported the justice of this proceeding by observing, art. 48—1,* that, by properly transposing the letters, the reasoning employed to deduce the value of a particular side, would have applied, word for word, to any other side.

Now this transposition of letters is equivalent to another mode of proceeding, that may be explained as follows.

We might regard each equation of m as deduced from a separate diagram: and the method of correlations would then require us to compare the diagram whence the first equation m was deduced, with the diagrams belonging to the three other equations.

Let us assume, for example, fig. 191—1 to be the diagram chosen as a standard, and whence the first equation m was deduced. Altering the order of the letters A, B, C, D, in such manner that B shall take the place of A, C of B, D of C, and, finally, A of D; the figure 191—1 changes into the figure 191—2; and the side b takes in the latter the situation held in the former by the side a.

Fig. 191.



Now the two figures differing only in the names of their sides, it will follow that whatever value is deduced from the first diagram for a, a similar value might be

* By an error of the press there are two articles 48; it is the first which is here referred to.

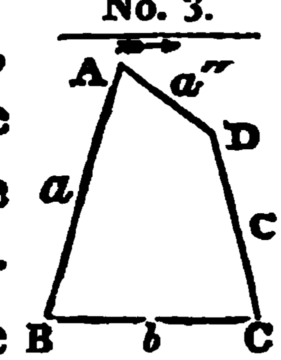
deduced from the second diagram for b; whilst a mere inspection of the figure shows the last of these values to result from the former by the transposition of letters used for that purpose in establishing the equations m.

The value of b, therefore, deduced from figure 191—2 is identical with the second equation m.

This value was deduced from the arrangement of points exhibited in figure 191—2, but it will apply, by the theory of correlation explained in the preceding pages, to any other form of the diagram.

Let us, then, imagine the points A, B, C, D, figure 191—2, to alter their arrangement until they attain such a disposition, fig. 191—3, that each line and angle in this new figure shall agree with the lines and angles of the same name in fig. 191—1. The expression obtained from fig. 191—2, for the line b, will apply to the value of that line in the diagram fig. 191—3, but the side b in this last diagram is the same with the side b of the diagram 191—1. And we must conclude, as we had already shown in art. 48—1, that fig. 191—1, is justly represented by the second equation m.

Fig. 191.
No. 3.



The remaining equations m may be deduced from the standard equation by similar reasoning.

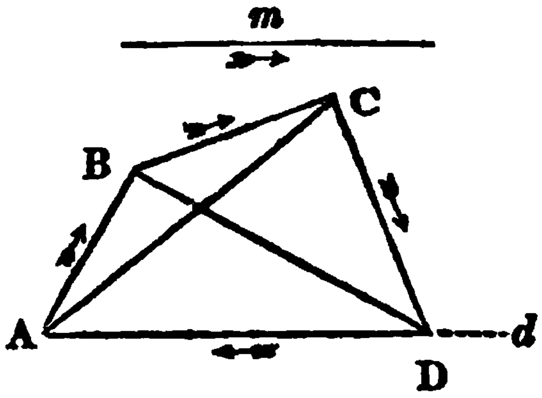
And it now remains for us to extend this process to other closed figures that are found in the relations of four points, and to deduce the equations of these figures from the standard equation hitherto employed.

The simplest relations of four points consist, we recollect, of distances, and the comparison of directions two and two.

We may present these relations graphically by exhi-

biting in one diagram, fig. 191, the directions and distances of all the points; and it has already been shown that a figure so constructed may be analyzed, art. 40, into four triangles and three closed figures of four sides.

Fig. 192.



Two of the latter are shown separately in figures 173 and 174.

In the first of these figures the sides follow the sequence of the letters ABCD.

In the second the sides and points follow a different sequence.

But to construct in this way a diagram that shall justly represent the case under investigation, it is necessary that we should know, in regard to that case, the position of the points A, B, C, D.

In what manner, then are we to proceed in estimating the sides and angles when, instead of the arrangement of the points being given, that arrangement is itself an object of research?

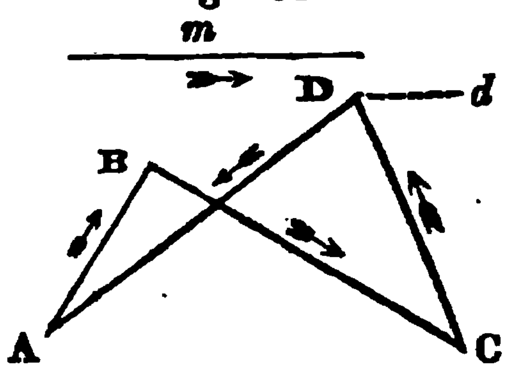
Whilst indeed we argue on one diagram only, the change of the equations, art. 108, will compensate any alterations that may be required in the figure: but if two diagrams are to be used simultaneously, the equations of the one must be rendered consistent with those of the other.

The obvious method of proceeding is to derive the equations from one of the two diagrams, and afterwards to estimate the changes which these equations undergo when the diagram whence they were derived is changed into the other.

In effecting this comparison, we remark, that until

the end of a geometrical investigation we rarely know how far the position we have assigned the points agrees with their real sequence.

Fig. 193.



The point C of fig. 192 might have held the place which has been assigned to D; and the figure 193, constructed by joining the points in the order of their sequence, would then have agreed in form with ACDB, a quadrilateral, found in fig. 192.

And, in like manner, causing B to take the place of D, C of B, and D of C, a figure constructed by joining the points in the order of their sequence would agree in form with ADBC, the remaining quadrilateral that results from the analysis of the figure in question.

To obtain the closed figures of three sides, we have merely to suppose two of the points A, B, C, D, coincident; and to reason in other respects as above.

From these remarks we learn not only that every closed figure included in the relations of four points is merely a particular form of a single figure taken as a standard, but further, that it is not until the end of the investigation that we can, in general, decide which among these forms is produced by writing the points, A, B, C, D, in the order of their sequence.

Arrived at this result, we can apply it to determine from the first equation *m*, the equations of the closed figure ABDC, fig. 192.

That closed figure, we recollect, is identical in form with 193; and is obtained from the latter by interchanging the letters D and C.

The first equation, *m*, we have also seen, applies equally to 193, and to ABCD, 192, both of which are

formed by uniting the points according to the sequence of the letters.

But since the interchange of C and D causes the lines BC and DA, of 193, to become the lines BD and CA of 192, the equation m will be rendered applicable to ABDC, 192, by writing b' in place of b, and a' in place of a'', or, that equation will become,

$$a = b' \cos. b'a + c \cos. ca + a' \cos. aa'.$$

And proceeding according to this method, and writing as follows:

$$\begin{array}{ccccccccc} A & - & B & - & C & - & D & - & A \;\ldots\ldots\; (n) \\ \vdots & & \vdots & & \vdots & & \vdots & & \\ a & & b & & c & & a'' & & \end{array}$$

the names of the points whose relations are sought, and of the distances between them; we can, by interchanging the letters, discover the alterations that must be made in the standard equation, in order to adapt it to other polygons.

Thus, shifting the letters until the series becomes,

$$\begin{array}{ccccccccc} B & - & C & - & A & - & D & - & B \;\ldots\ldots\; (p) \\ \vdots & & \vdots & & \vdots & & \vdots & & \\ b & & a' & & a'' & & b' & & \end{array}$$

we observe that in the equation

$$a = b \cos. ab + c \cos. ac + a'' \cos. a''a$$

a must be changed into b, b into a', c into a'' and a'' into b', when there arises,

$$b = a' \cos. ba' + a'' \cos. a''b + b' \cos. bb',$$

an equation which gives the value of b, in the polygon ACBD, fig. 192.

The closed figures that arise from the analysis of four points, fig. 192, are obtained, art. 40, by uniting point with point, according to such an arrangement that, com-

2 H

mencing with any one, and passing through all the remainder, we may return without visiting any point twice, to the place where the route commenced.

The order according to which the points are visited is obtained from the series (n).

And hence, to discover all the closed figures that are formed in the relations of four points, we should successively make in the series (n), every possible interchange of letters that is consistent with the first letter remaining the same as the last.

The investigation we have pursued has led to an easy method of deducing, from a common standard, the equations which arise in analysing the relations of four points; nor can it be doubted, after what we have shown respecting the correlation of figures, that such changes will occur in the equations as are required by the changes in the type, and that, consequently, the equations we obtain by this rule will be simultaneous.

It must not be forgotten, however, that certain conventions were necessary in order to produce this correspondence between the changes of the equations and those of the figure, and it will be useful for us to trace the application of these conventions, in the problem now under consideration.

The first case that will attract our attention relates to the equations obtained for different sides of the same figure.

In each of these equations, one or more angles will be found in common, and we have to show that, notwithstanding the alterations in the directions of the sides, occasioned by an interchange of letters, the angles alluded to will still be identical.

This fact will be rendered evident by inspecting the

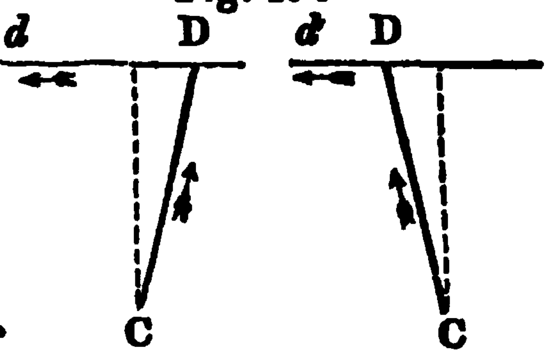
figures 191—1, and 191—3, where the inclination of b and a, the only angle common to the formula derived from those figures, is measured in the former by BCc, and in the latter by the equal angle ABC.

A similar result will not, however, be deduced from a comparison of the figures 193 and 192.

Comparing the polygon ABCD of the former, with the polygon of the same name in the latter, we find the line CD holding contrary directions—its projection upon m is, therefore, reversed—and, according to the rule in art. 108, a change must take place in the signs.

But when a line is projected upon another, to change the sign of the projection is equivalent, art. 107, to changing the angle which measures the inclination of these lines into the supplement of that angle: a result that we observe in figure 194, where, by a change in the projection of c, the inclination of c and m is changed from CDd to CDd'.

Fig. 194.



The angle which measures the inclination of c and m, in the polygon 193, ought, therefore, to be the supplement of the corresponding angle in the polygon ABCD, 192; and, accordingly, we find this relation in the angles CDd of those two figures.

The polygon ABDCA, figure 192, is passed over in the same sequence with the polygon 193, and what has been said of one will apply to the other.

It results, therefore, from our inquiry, that in analysing the relations of A, B, C and D, into two polygons, ABCDA, and ABDCA, figure 192, the angle ca of the one will be the supplement of ca in the other: and, accordingly, when the equations deduced from the first

polygon are used in conjunction with the equations deduced from the second, we must change, in the latter, the sign of every term that is multiplied into cos. *ca*.

The change of direction that has rendered necessary this alteration in the signs, is pointed out in the series used, in the present article, to obtain the equations of the one polygon from those of the other.

For, writing the series which gives the first of these two polygons,

$$\begin{array}{ccccccccc} A & \ldots & B & \ldots & C & \ldots & D & \ldots & A \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ & a & & b & & c & & a'' & \end{array}$$

and comparing it with the series belonging to the second,

$$\begin{array}{ccccccccc} A & \ldots & B & \ldots & D & \ldots & C & \ldots & A \\ & \vdots & & \vdots & & \vdots & & \vdots & \\ & a & & b' & & c & & a' & \end{array}$$

we observe, that in reading one of those arrangements we pass from C to D, whilst in reading the other that sequence is reversed.

114. The reasoning here employed on the relations of four points will apply to those of any number.

And, combining what is said respecting the change of signs, with the method developed in the beginning of the preceding article, we readily deduce the equations which arise in analysing the relations of any number of points.

These equations, it will be recollected, are developed in succession from a single equation taken as a standard; but as the method we have followed will frequently reproduce the same expression under different forms, it is necessary to guard against this superfluity.

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The considerations that enable us to do so are obvious.

The equations in question are divided into sets, each set arising from the analysis of a different closed figure.

To avoid repetitions, therefore, we must so proceed in uniting point with point, that we shall not pass "twice" over the same closed figure.

Thus to analyse the relations of n points into polygons of as many sides:

Commencing with the point A, we have $n-1$ points remaining, any one of which may be united with A; that is, A occupying the first place in the series, art. 113, any one of the remaining letters may be put in the second place. Let us suppose C the point so united with A, we have then to connect C with any one of the points remaining, that is, with any of the n points except A and C.

In our series, therefore, we may put A in the first place, either of the $n-1$ remaining letters in the second place, and either of the $n-2$ letters, which then remain, in the third. And the number of ways of doing so will be $(n-1)\,.\,(n-2)$.

But this solution made, and omitting the three points already united, we have still a choice among $n-3$ others.

And continuing the process, and combining the number of selections at each step, the total number will be $(n-1)\,(n-2)\,(n-3) \ldots\ldots 3.\,2.\,1.$

This process, whilst it avoids the repetitions that would arise from an unmethodical development of the polygons, does not prevent our passing over the same figure in different directions, and thus estimating it as two polygons; but the superfluous series, obtained in this way, will be easily recognized, since, read backwards, they

agree with series previously deduced: the fact serves, however, to reduce the number of distinct polygons to $(n-1)\,(n-2)\,(n-3)\ldots\ldots 3$.

Each of these figures produces n distinct equations, and the number of the latter obtained from an analysis by closed figures of n sides, will therefore be $n\,(n-1)\,(n-2)\,(n-3)\ldots\ldots 3$, or, half the number of permutations that can be made out of n things taken all at a time.

When, by the preceding rules, we have determined the several closed figures into which the relations of n points can be analysed, the n equations resulting from each are best deduced by the rule in art. 113.

The process we have employed will also serve to determine the polygons of any number of sides less than n.

To discover, for example, the polygons of $n-1$ sides, we must, in the first place, form new series, by omitting, first A, then B, and so on successively, until we have written down n distinct combinations of $n-1$ points.

Secondly, applying to these combinations the method already explained, we form from each $(n-2)\,(n-3)\,(n-4)\ldots\ldots 3$, distinct closed figures. And hence, the number of such figures formed from all the combinations will be $n\,(n-2)\,(n-3)\ldots\ldots 3$. And as every such figure produces $n-1$ equations, the total number of the latter will be $n\,(n-1)\,(n-2)\ldots\ldots 3$.

In condensing these observations into a practical rule, we shall make $n=4$, with the view of avoiding the great length to which the process extends when the points are more numerous.

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RULES FOR ANALYSING THE RELATIONS OF *n* POINTS.

Illustrated by the case where n is 4.

1. *Write in a series commencing and ending with A, and arranged according to the order of the alphabet, the names of the points, placing between every two of these, but lower down, the name of the line that unites those poiuts,*

$$\left.\begin{array}{ccccccccc} \mathrm{A} & & \mathrm{B} & & \mathrm{C} & & \mathrm{D} & & \mathrm{A} \\ & a & & b & & c & & a'' & \end{array}\right\} \quad (1)$$

2. *Keeping* A *in the first and last place, make all possible permutations of the other letters, omitting, however, those series which, read backwards, agree with series already obtained.*

$$\left.\begin{array}{ccccccccc} \mathrm{A} & & \mathrm{C} & & \mathrm{B} & & \mathrm{D} & & \mathrm{A} \\ & a' & & \overline{b} & & b' & & a'' & \end{array}\right\} \quad (2)$$

$$\left.\begin{array}{ccccccccc} \mathrm{A} & & \mathrm{D} & & \mathrm{B} & & \mathrm{C} & & \mathrm{A} \\ & \overline{a''} & & \overline{b'} & & b & & \overline{a'} & \end{array}\right\} \quad (3)$$

3. *Verify the correctness of the operation, by observing whether the number of series obtained in this way, including the type, is equal to* $(n—1)$, $(n—2)$, $(n—3)$ . . . . . 5. 4. 3.

4. *Commencing with the first series, read them all in succession, proceeding from left to right.*

Observe where, in passing from one letter to another, the order of the alphabet is violated.

Over the name of the line uniting the points where this happens place a negative sign.

5. *Apply to each polygon, so developed, that is, to each series of italic letters, the process, art.* 112, *theo.* 9,

form 2; making each term, successively, the left hand number of the equation—and writing a negative sign before those cosines that involve "one" letter having the negative sign above it.

$$\left.\begin{aligned} a &= +\, b\cos.ab + c\cos.ac + a''\cos.aa'' \\ b &= +\, c\cos.bc + a''\cos.ba'' + a\cos.ba \\ c &= +\, a''\cos.ca'' + a\cos.ca + b\cos.cb \\ a'' &= +\, a\cos.a''a + b\cos.a''b + c\cos.a''c \end{aligned}\right\}\ \begin{matrix}\text{From}\\\text{polygon}\\1.\end{matrix}$$

$$\left.\begin{aligned} a' &= -\, b\cos.a'b + b'\cos.a'b' + a''\cos.a'a'' \\ -b &= +\, b'\cos.bb' + a''\cos.ba'' + a'\cos.ba' \\ b' &= +\, a''\cos.a''b + a'\cos.b'a' - b\cos.b'b \\ a'' &= +\, a'\cos.a''a' - b\cos.a''b + b'\cos.b''b' \end{aligned}\right\}\ 2.$$

$$\left.\begin{aligned} a'' &= +\, b'\cos.b'a'' - b\cos.a''b + a'\cos.a''a' \\ b' &= -\, b\cos.b'b + a'\cos.b'a' + a''\cos.b'a'' \\ -b &= +\, a'\cos.b'a + a''\cos.ba'' + b'\cos.bb' \\ a' &= +\, a''\cos.a'a'' + b'\cos.a'b' - b\cos.a'b \end{aligned}\right\}\ 3.$$

When these equations are not sufficient, we must proceed to analyse by polygons of $n-1$ sides.

RULES FOR ANALYSING BY POLYGONS OF $n-1$ SIDES.

1. *Write all the combinations that can be made out of the names of the points taken $n-1$ at a time.*

A	B	C	A
A	B	D	A . . . a
A	C	D	A
B	C	D	B

2. *Proceed with each of these series as with the series* (1), *that served as a type,*

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$$\begin{array}{ccccccccl}
A & & B & & C & & A & & \\
 & a & & b & & \overline{a'} & & \ldots\ldots & (4)
\end{array}$$

$$\begin{array}{ccccccccl}
A & & B & & D & & A & & \\
 & a & & b' & & a'' & & \ldots\ldots & (5)
\end{array}$$

$$\begin{array}{ccccccccl}
A & & C & & D & & A & & \\
 & a' & & c & & a'' & & \ldots\ldots & (6)
\end{array}$$

$$\begin{array}{ccccccccl}
B & & C & & D & & B & & \\
 & b & & c & & \overline{b'} & & \ldots\ldots & (7)
\end{array}$$

3. *Verify the operation by observing whether each of the combinations, α, produces (n—2) (n—3) . . . 4. 3, distinct series*.*

4. *Proceed to deduce the equations as in the preceding rule.*

$$\left.\begin{array}{rl}
a &= +b\cos.ab - a'\cos.aa' \\
b &= -a'\cos.ba' + a\cos.ba \\
-a' &= +a\cos.a'a + b\cos.a'b
\end{array}\right\}\text{From polygon } 4.$$

$$\left.\begin{array}{rl}
a &= +b'\cos.ab' + a''\cos.aa'' \\
b' &= +a''\cos.b'a'' + a\cos.b'a \\
a'' &= +a\cos.a''a + b'\cos.a''b'
\end{array}\right\}\qquad 5.$$

$$\left.\begin{array}{rl}
a' &= +c\cos.a'c + a''\cos.a'a'' \\
c &= +a''\cos.ca'' + a'\cos.ca' \\
a'' &= +a'\cos.a''a' + c\cos.a''c
\end{array}\right\}\qquad 6.$$

$$\left.\begin{array}{rl}
b &= +c\cos.bc - b'\cos.bb' \\
c &= -b'\cos.cb' + b\cos.cb \\
-b' &= +b\cos.b'b + c\cos.b'c
\end{array}\right\}\qquad 7.$$

* In the example used to illustrate this rule, $n = 4$; and the first term of the series (n—2) (n—3) . . 4. 3, would be less than the last term, 3, at which we are to stop : to discover how this happens, we have only to recollect that our series occurred under the form $\frac{(n-2)\,(n-3)\,.\,.\,4.\,3.\,2.\,1}{2}$ and which for $n = 4$, becomes $\frac{2.1}{2} = 1$.

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Should these equations still prove insufficient to discover the particular relations sought, we may form others, by extending the process already used, to figures of $n-2$ sides; and, finally, to triangles.

We have now only to remark, that instead of the second form of theorem 9, we may substitute, in the preceding rule, any of the forms given in art. 112; the wording, in all other respects, remaining the same: the fourth form, when signs are used in place of cosines, is especially to be preferred when the number of points is few.

It is not, however, necessary, in any case, to use all the equations that would result in this way; their number, when the points are numerous, is immense; and those only must be selected that are required by the nature of the problem; in what manner to conduct this selection will be the subject of a separate inquiry.

PRELIMINARY REFLECTIONS TO SECTION IV.

All closed figures being compared with the right angled triangle, the relations of the latter should be expressed with the utmost possible simplicity: we have not, hitherto, formed a simple, or even a manageable expression, for the ratio of the sides, in terms of the angles. How is the deficiency to be supplied? The best substitute seems to be a table. A table may be formed by assuming an angle, as small as any that we have occasion to use, and by calculating the ratios in question for every whole multiple of this small angle.

INQUIRIES SUGGESTED BY THESE REFLECTIONS.

Relations of angles about a point. Numerical value of the sine of a small angle: simple method of obtaining by consecutive calculations the sines, cosines, &c. of every multiple of this angle.

SECTION IV.

RELATIONS WHICH THE ANGLES OF THE TYPE BEAR TO THE RATIOS OF ITS SIDES.

Measurement of angles, and notation used to express their quantity—possibility of reducing the sine of any angle to that of a small aliquot part of it—the other ratios may be obtained from the sine—the sine of a particular angle found—calculation of the sine of a very small angle—and thence of all the ratios of any angle whatever—limits of the ratios—their algebraic signs.

115. In articles 42 and 79, plane angles were shown to be definite parts of the infinite plane space that surrounds a point.

This space forms a natural, and, often, a convenient unit by which to measure such angles.

When, however, this "largest" angle is taken as our unit, all other angles will be proper fractions.

And as fractions are less simple expressions than whole numbers, it is often convenient, in practice, to assume the unit an angle so small that all other angles shall be whole multiples of it.

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Art. 115. Measurement of angles and notation used to express their quantity.

This supposes the unit to be the least angle that is used.

In very accurate tables, the least angle that is used is called a *second*, and is the $\frac{1}{1.296.000}$ part of the plane space about a point.

An angle formed by the juxta-position of sixty such small angles, and, therefore, equal to sixty seconds, is called a *minute:* and an angle equal to sixty minutes is a *degree.*

Hence, if the second were alone denoted by the *unit of number*, the minute would be denoted by 60, and the degree by 3600.

But following the principle adopted in arithmetic, where the denominations of tens, hundreds, thousands, &c. have each their unit, distinguished by the place it occupies; geometers have, in like manner, represented one second, one minute, one degree, by the unit of number; distinguishing the several denominations apart by *accents* placed over them.

A second has two accents, thus, $1''$; a minute, one, $1'$, and a degree is without accents, which last agreement is often expressed by placing a cypher above the degree, in the place occupied by the accents of the other denominations, thus, 1°.

But although the plane space about a point is in many respects the natural unit of angles, these quantities, as we have seen, art. 110, may be increased without limit; and, on the other hand, they admit an indefinite subdivision. An angle, therefore, can only be the "smallest," with relation to the degree of accuracy which we decide upon using: in this way, the angle that is considered as the

unit, is sometimes a *third*, or the sixtieth part of a second; or even a *fourth*, which is the sixtieth part of a third; denominations whose units are distinguished by three and four accents, thus, $1'''$, a third, $1''''$, a fourth; the unit of number, when not provided with such accents, representing the whole of plane space.

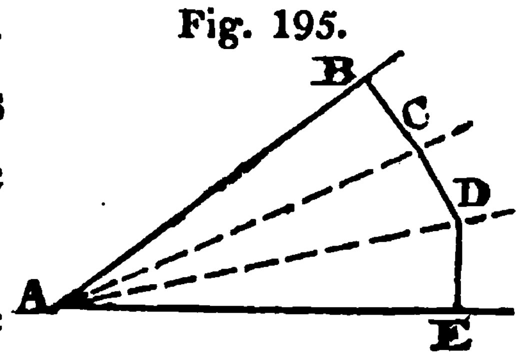
116. Choosing one of these small angles, the minute, for example, as the ultimate unit, and neglecting all smaller parts, we may regard every angle as formed of such integral parts placed in juxta-position; and, perhaps, in this way deduce the sine, cosine, &c. of the given angle from those of the unit.

If such a deduction is attainable, it will furnish us with the means of calculating the important tables proposed in art. 52; or, rather, it will cause this calculation to depend on that of the sine, cosine, &c. of a very small angle.

But it is only necessary to glance at the method used in the article alluded to, in order to perceive the possibility of that deduction which we seek to establish.

For drawing the perpendiculars mentioned in art. 52, and, for greater simplicity, choosing the foot of one perpendicular as the commencement of the next, we form a closed figure, ABCDEA. In this figure we may assume $AB = 1$, and we shall then have,

Fig. 195.



Chap. II. Of the elements to which place is referred.

Art. 116. Possibility of reducing the sine of any angle to that of a small aliquot part of it.

$$\begin{array}{ll} AB = 1 & a = 1 \\ AC = \cos.\ 1' & b = \sin.\ 1' \\ AD = \cos.^2 1' & c = \cos.\ 1' \sin.\ 1' \\ AE = \cos.^3 1' & d = \cos.^2 1' \sin.\ 1 \\ & a''' = \cos.^3 1'. \end{array}$$

We have chosen an angle which contains the unit three times, but the reasoning is evidently general, and we may, therefore, conclude, that by proceeding in the same way with any angle, the sides of the resulting closed figure would all depend upon the sine and cosine of the unit.

It is also evident from the construction itself, that all figures formed of the given sides, and with the given angle A, will be identical, and hence the data, we conclude, are such as suffice to determine the problem.

But, with sufficient data, the analysis of closed figures, established in the preceding sections, will afford us equations whence all parts of the figure can be determined.

The sine of A is not only a part of the figure but enters as a term in the equations alluded to.

The sine of A can therefore be expressed by means of the data, or, from what we have seen respecting the values of the sides, the sine of A can be expressed in terms of sine $1'$ and cosine $1'$; or, lastly, in terms of sine $1'$ alone, since the sine and cosine will presently be shown to depend upon each other.

117. The possibility of calculating, and forming tables of the sines of all angles, is, by this preliminary inquiry, reduced to the possibility of calculating the sine, either of $1'$, or of any other very small angle.

But since an equation can be deduced that expresses

sine A in terms of sine 1′, conversely, that equation will give the value of sine 1′ in terms of sine A.

118. And thus the possibility of calculating the tables is reduced to that of finding the sine of some one angle A.

Turning our attention towards this object, we observe, that in any right angled triangle whose hypothenuse is a,

$$\frac{a'}{a} = \cos.\ aa' \ldots\ldots\ldots\ldots 1$$

$$\frac{b}{a} = \cos.\ ba. \ldots\ldots\ldots\ldots 2 \ldots a$$

$$a = a' \cos.\ aa' + b \cos.\ ab \ldots\ldots 3$$

$$\tfrac{1}{4} = (aa') + (ab) \ldots\ldots\ldots 4$$

and as we have here four equations between five unknown quantities, it will follow, that provided we can find a fifth condition connecting the angles with the sides, the problem we seek, namely to find the sine of some one angle, will be completely resolved.

Such a condition will be obtained by assuming $b = a'$; for as the left hand numbers of the two first equations are then equal, the cosines of $(a'a)$ and (ba), and consequently the angles themselves, must be so likewise.

But with these substitutions the two last equations become

$$1 = 2 \cos.^2 aa' = 2 \sin.^2 aa'$$

$$\tfrac{1}{4} = 2\,(aa')$$

whence we have

$$\sin.\ \tfrac{1}{8} = \cos.\ \tfrac{1}{8} = \sqrt{\tfrac{1}{2}} = \tfrac{1}{2}\sqrt{2} = .7071068.$$

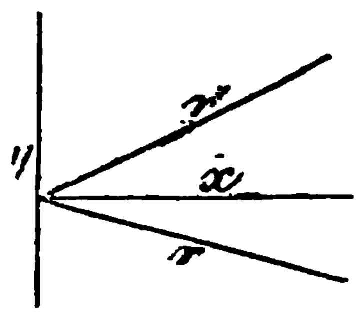
119. The next step in the investigation is to determine the sine of 1′ from this of $\frac{1}{8}$.

Chap. II. Of the elements to which place is referred.
Art. 119. Calculation of the sine of a very small angle.

And for that purpose, as we have already remarked, it will be necessary to establish an equation between sine A and sine of 1′. Or, which is a more general case of the same problem, between sine A and sine a, where a is an integral part of A.

To solve this question let us have recourse to article 100, equation 20.

Fig. 196.



Assuming the lines r' and r to lie as in figure 196; observing that, zr, zr' are each equal $\frac{1}{4}$, and consequently cos. zr and cos. zr' equal o; that $r'y = \frac{1}{4} - r'x$, and $ry = \frac{1}{4} + rx$; the equation becomes

$$\cos. rr' = \cos. r'x \cos. rx - \sin. r'x \sin. rx.$$

The angle rr' is the sum of the other two, and, hence, putting $r'x = a$, and $rx = b$, we have the theorem

$$\cos. (a + b) = \cos. a \cos. b - \sin. a \sin. b. \quad . \; . \; . \; \beta.$$

This equation will readily enable us to solve the question with a view to which it was introduced, namely, to express cos. A in terms of cos. a; where the smaller angle a is an integral part of the larger.

For taking in the first place $A = 2a$, and assuming $b = a$, the formula β becomes

$$\cos. A = \cos. 2a = \cos.^2 a - \sin.^2 a.$$

This value of A contains, it is true, both the sine and cosine of a; but substituting in the third of the equations α, the values of a' and b obtained from the two first, we have

$$1 = \cos.^2 aa' + \cos.^2 ba;$$

or, as cos. ab is the sine of aa',

$$1 = \cos.^2 aa' + \sin.^2 aa';$$

a result that may be written more conveniently under the form

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$$1 = \cos.^2 a + \sin.^2 a.$$

Hence, deducing the value

$$\cos.^2 a = 1 - \sin.^2 a,$$

and substituting it in the formula for cos. A, that formula becomes

$$\cos. A. = \cos. 2\,a = 1 - 2\sin.^2 a \;.\;.\;.\;.\; \gamma$$

where, as cos. A is expressed altogether in terms of sin. a, we can, if required, find sin. A in terms of the same quantity.

The reasoning employed upon the case where $A = 2a$, would lead us to a general solution of the problem; but for the object immediately in view the particular case already solved is abundantly sufficient.

That object we recollect is to find the sine of a very small angle.

Now the equation γ, which expresses cos. A in terms of sine a, will, conversely, give the sine of the latter in terms of the cosine of the former, or,

$$\sin. a = \sqrt{\{\tfrac{1}{2} - \tfrac{1}{2}\cos. 2a\}}$$

which, putting $2a = A$, may be written

$$\sin. \tfrac{1}{2} A = \sqrt{\{\tfrac{1}{2} - \tfrac{1}{2}\cos. A\}}$$

Whence, if cos. A is known, sin. $\frac{1}{2}$ A is known, and since $\cos.^2 = 1 - \sin.^2$, the cos. $\frac{1}{2}$ A is also known.

But, repeating these operations, we shall have

$$\sin. \tfrac{1}{4} A = \sqrt{\{\tfrac{1}{2} - \tfrac{1}{2}\cos. \tfrac{1}{2} A\}}$$
$$\cos. \tfrac{1}{4} A = \sqrt{\{1 - \sin.^2 \tfrac{1}{4} A\}}$$
$$\sin. \tfrac{1}{8} A = \sqrt{\{\tfrac{1}{2} - \tfrac{1}{2}\cos. \tfrac{1}{4} A\}}$$
$$\cos. \tfrac{1}{8} A = \sqrt{\{1 - \sin.^2 \tfrac{1}{8} A\}}$$
&c. . . . . &c.

As this process may be continued to any extent, it appears that when cos. A is known, we can find by suc-

cessive operations the sine of $\frac{A}{2^n}$; where n may be taken as great, and consequently $\frac{A}{2^n}$ as small, as we please.

In the particular case where $A = \frac{1}{8}$, we have found the cosine of A to be .7071068; and, hence, whatever value is assigned to n, the successive operations described will readily lead us to the sine of $\frac{1}{8.2^n}$ or, which is the same thing, to the sine of $\frac{1}{2^n}$.

The angle $1'$ does not happen to fall within the form $\frac{1}{2^n}$; but when n is taken very great, the sines, as they are successively computed, will be found to approach towards a constant number multiplied into the angle; or, in other words, the sine divided by the angle will approach a constant limit.

The accuracy of the tables must be determined by their size.

And hence, when the difference between the limit in question and the ratio of the sine and angle is not found in the tables, we may regard that limit as rigorously attained.

This will happen for some particular value n' of n; and, taking n' greater than this value, and assuming

$$a < \frac{1}{2n'},$$

we have,

$$\frac{\sin.\ a}{a} = c$$

where c is the constant limit.

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Art. 119. Calculation of the sine of a very small angle.

The number c, though constant for the same unit, varies with the latter; and provided the angle assumed as unity is less than the whole of plane space divided by $2^{n'}$, we shall have,

$$\sin.\ 1 = c$$

where 1 is the small angle we have assumed as unity.

Hence, supposing the tables not carried beyond a certain degree of accuracy, seven or eight places of figures, for example, it will appear, on performing the processes described in this article, that one minute is less than the whole of plane space divided by $2^{n'}$; whence,

$$\sin.\ 1' = c.$$

120. The value of c will, in this case, be .0002909, and the analysis given in art. 116 requires that we should now investigate a method of determining from the sine of 1′, that of all angles which are whole multiples of this unit.

The formula

$$\cos.\ (a + b) = \cos.\ a \cos.\ b - \sin.\ a \sin.\ b$$

will readily afford us the means of so doing. For, assuming $a = a'$, and $b = \frac{1}{4} + b'$, that equation becomes,

$$\cos.\ (\tfrac{1}{4} + a' + b') = -\sin.\ a' \cos.\ b' - \cos.\ a' \sin.\ b'$$

or,

$$\sin.\ (a' + b') = \sin.\ a' \cos.\ b' + \cos.\ a' \sin.\ b';$$

where, assuming $a' = 1'$, and b', successively, equal to $1'$, $2'$, $3'$, &c. we obtain,

$$\sin.\ 2' = 2 \cos.\ 1' \sin.\ 1'$$

$$\sin.\ 3' = \sin.\ 1' \cos.\ 2' + \cos.\ 1' \sin.\ 2'$$

$$\sin.\ 4' = \sin.\ 1' \cos.\ 3' + \cos.\ 1' \sin.\ 3'$$

&c. . . . . &c.

In tables of the extent here supposed, the cosine of 1′ may be taken equal to unity; and these equations will then assume a form rather more simple.

The process may be continued as far as we please; and since the sine of an angle is thus expressed in terms of the sine and cosine of the preceding angle, and of the known sine of 1′, it is evident, that, by successive operations, we can calculate the sine of every whole multiple of 1′.

The sine has been chosen, in these calculations, as affording greater facility than any of the other ratios: but when a complete table of any one ratio has been obtained, the remaining five can be readily deduced from it.

The cosine can be obtained from the sine by means of the equation, art. 120,

$$\cos.^2 + \sin.^2 = 1.$$

And to find the tangent, we may proceed as follows,— in any right angled triangle,

$$\frac{b}{a} = \sin.\ a'a$$

$$\frac{a'}{a} = \cos.\ a'a$$

$$\frac{b}{a'} = \tan.\ a'a$$

hence, dividing the first equation by the second,

$$\frac{\sin.\ a'a}{\cos.\ a'a} = \tan.\ a'a$$

or, as we may express the result more generally,

$$\frac{\sin.}{\cos.} = \tan.$$

The three remaining ratios are the reciprocals of the sine, cosine and tangent; and thus, when the sines and cosines have been calculated, the other four ratios can be found from them, by division.

The particular arrangements adopted in the tables of these ratios do not belong to a work of this nature; but it will be proper to mention, that, in most calculations, the logarithms of these ratios are usually employed, instead of the ratios themselves; and that, to avoid the frequent occurrence of negative logarithms, it has been usual to set down in the table, not the logarithm itself, but that logarithm increased by 10. The convenience of this practice may be greatly doubted; the final rejection of 10 for every logarithm used, renders both the formula and the practice unnecessarily complicated, and could have been avoided by placing in the tables the arithmetical complements of the negative logarithms, with the negative characteristics increased by unity.

A few remarks on the limits of the six ratios, and on their algebraic signs, will complete all that we need say upon this subject.

121. When the angles are very small, the limits to which the sines, cosines, &c. approach, are as follows:

sine 0	secant 1
cosine . . 1	cosecant . . . ∞
tangent . . 0	cotangent . . ∞.

The sine, divided by the angle, we have already seen, approaches a certain constant number, which is also the limit of the tangent divided by the angle; the value of this limit, to seven places of figures, and cor-

responding to the case where 1′ is the unit of angles, has been given in the preceding article: but, except for the purpose of calculating the tables, the most convenient unit of angles is the plane space about a point; accordingly, it is with respect to this unit that the limit in question most frequently occurs: it is denoted, in this work, by the symbol 2π and is equal to 6.2883185.

122. The algebraic signs of the ratios will be readily deduced from the principles already laid down regarding that subject; for since, representing the whole of plane space by unity, we have,

$$\begin{aligned} \sin.\left(\tfrac{1}{4}+a\right) &= \cos. a, \\ \sin.\left(\tfrac{1}{2}+a\right) &= -\sin. a, \\ \sin.\left(\tfrac{3}{4}+a\right) &= -\cos. a, \end{aligned}$$

it follows that,

Sin. a is positive when a is less than ½, and negative when a lies between ½ and 1.

Or, extending the result to angles that are greater than unity,

Sin. a is positive when a lies between an even and an odd multiple of ½, and negative when it lies between an odd and even multiple of ½.

A similar investigation applied to the cosines will demonstrate that,

Cos. a is positive when a lies between 0 and ¼, or ¾ and 1, and negative when it lies between the limits ¼ and ½, or ½ and ¾.

Or, more generally,

Cos. a is positive when a is greater than $(4m-1)\cdot\tfrac{1}{4}$ *and less than* $(4m+1)\cdot\tfrac{1}{4}$, *and negative when a is*

greater than $(4\,m + 1)\,.\,\frac{1}{4}$ *and less than* $(4\,m + 3)\,.\,\frac{1}{4}$, *where m is any whole number.*

The tangent, being equal to the sine divided by the cosine, will be positive when the sine and cosine have the same algebraic sign, and negative when their algebraic signs are opposite : or,

Tan. a will be positive when a lies between an even and an odd multiple of $\frac{1}{4}$, and negative when it lies between an odd and an even multiple.

The secant, cosecant and cotangent will have the same algebraic signs as the cosine, sine and tangent, respectively.

PART III.

ANALYSIS OF PARTICULAR PROBLEMS.

PRELIMINARY REFLECTIONS.

The preceding analysis referring wholly to the relations of points, its further development may naturally be divided into distinct propositions, according as the relations of three, four, or a greater number of points are inquired into.

The relations of two points have been fully developed.

A single point has no relations.

And, yet, if we suppose angles formed about a point, these relations would include all the relations of direction that could occur in any of the propositions just mentioned. For supposing in space any number of simple directions, and that lines are drawn parallel to them from a given point, the angles formed by these lines will be equal to the angles formed by the directions to which they are parallel.

In treating of such angles, as many distinct divisions can be made, as in treating of the relations of points.

ARRANGEMENTS SUGGESTED BY THESE REFLECTIONS.

First Division. Relations of the angles about a point. *Subdivisions*. Dependent on the number of divergent lines.

Second Division. Relations of any number of points

in space. *Subdivisions.* Dependent on the number of points.

These general principles of arrangement admit of modification whenever principles of a more partial kind tend to further—the sole object of classification—the ready acquirement and use of knowledge.

Such a subordinate principle arises from the facility with which graphic models can be delineated on plane surfaces; and hence the angles formed by divergent lines that lie in one plane will form the subject of a separate section.

CHAPTER I.

DETAILED ANALYSIS OF THE RELATIONS OF DIRECTION; AND OF THE RELATIONS PECULIAR TO THREE—TO FOUR —AND TO A GREATER NUMBER OF POINTS.

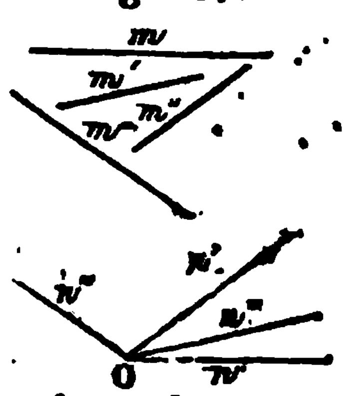
SECTION I.

RELATIONS OF THREE DIVERGENT LINES THAT LIE IN ONE PLANE.

The relations of direction are the same with the relations of angles that are formed about a common point—relations of three directions that lie in one plane—converse relations of the type of closed figures—transformations of the converse relations—tables of the most useful relations of angles about a point and in one plane.

123. Straight lines, m, m', m'', &c., whose lengths are not made an object of inquiry, may be regarded as "simple" directions.

Fig. 197.



Their only relations will be their "relative" directions, or the angles which they mutually form, and it is to these that we have now to turn our attention.

Assuming at pleasure a point O, and drawing from it straight lines parallel respectively to m, to m', to m'', &c., the inclinations of these last will, art. 71, be those of the former lines; and our inquiry is therefore reduced to an analysis of the relations of angles about a point.

124. The simplest case of the problems we are con-

sidering is, that which involves only three directions, and where these directions are in one plane.

Let the directions be m, m' and m''; and from any point, O, draw parallel to them the lines n, n' and n'': these last will be also in one plane, art. 69—5 and 63—5.*

Let n' be that direction which lies between the other two—we have

$$(nn'') = (nn') + (n'n'');$$

or, calling these angles, a, b and c

$$(1) \ldots\ldots\ldots a = b + c.$$

Hence, by articles 19 and 20, we have

$$(2) \ldots \cos. a = \cos. b \cos. c - \sin. b \sin. c = \cos. (b + c)$$
$$(3) \ldots \sin. a = \sin. b \cos. c + \cos. b \sin. c = \sin. (b + c)$$

Dividing the second of these equations by the first,

$$\frac{\sin. a}{\cos. a} = \frac{\sin. b \cos. c + \cos. b \sin. c}{\cos. b \cos. c - \sin. b \sin. c} = \frac{\sin. (b + c)}{\cos. (b + c)};$$

or, dividing numerator and denominator of the second fraction by $\cos. b \cos. c$, and recollecting that $\frac{\sin.}{\cos.}$ is equal to tangent;

$$(4) \ldots \tan. a = \frac{\tan. b + \tan. c}{1 - \tan. b \tan. c} = \tan. (b + c)$$

The reciprocals of these expressions will be, respectively, sec. a, cosec. a, and cotangent a.

From equation 1 we obtain

$$(5) \ldots\ldots b = a - c;$$

and, consequently, the sine, cosine, &c., of b may be

* By art. 69—5 it may be shown that n, n' and n'' are at right angles to a common line, whence, art. 63—5, they are in one plane.

deduced from those of a by substituting a for b, and $-c$ for $+c$; making these substitutions, and observing, art. 122, that sin. $(-c)$ is equal to $-$ sin. c, there arises

$$(6)\ ..\ \cos. b = \cos. a \cos. c + \sin. a \sin. c = \cos. (a - c)$$

$$(7)\ ..\ \sin. b = \sin. a \cos. c - \cos. a \sin. c = \sin. (a - c)$$

$$(8)\ ..\ \tan. b = \frac{\tan. a - \tan. c}{1 + \tan. a \tan. c} = \tan. (a - c)$$

The equations

$$\sin. (a + b) = \sin. a \cos. b + \sin. b \cos. a$$

$$\sin. (a - b) = \sin. a \cos. b - \sin. b \cos. a$$

which are the same with 3 and 7, give, by addition and subtraction,

$$(9)\ ...\ \sin. a \cos. b = \tfrac{1}{2} \{\sin. (a + b) + \sin. (a - b)\}$$

$$(10).\ ..\ \sin. b \cos. a = \tfrac{1}{2} \{\sin. (a + b) - \sin. (a - b)\}$$

And by the same process we obtain from the equations

$$\cos. (a + b) = \cos. a \cos. b - \sin. a \sin. b$$

$$\cos. (a - b) = \cos. a \cos. b + \sin. a \sin. b$$

the following

$$(11).\ ..\ \cos. a \cos. b = \tfrac{1}{2} \{\cos. (a + b) + \cos. (a - b)\}$$

$$(12).\ ..\ \sin. a \sin. b = \tfrac{1}{2} \{\cos. (a - b) - \cos. (a + b)\}$$

These transformations, besides the practice which they afford the student, are of frequent use in analysis, and lead to other forms that we shall have occasion to employ.

One obvious change that can be made in them is produced by substituting single angles for those which are compound. Thus assuming

$$a + b = \alpha, \text{ and } a - b = \beta$$

the equations 9, 10, 11 and 12 become, respectively,

$$(13)\ ...\ \sin. \frac{\alpha + \beta}{2} \cos. \frac{(\alpha - \beta)}{2} = \tfrac{1}{2} \{\sin. \alpha + \sin. \beta\}$$

Chap. I. Detailed analysis of the relations of direction; and of the relations peculiar to three, to four, and to a greater number of points.

Art. 124. Relations of three directions that lie in one plane.

$$(14) \quad . \; . \; . \quad \sin.\frac{\alpha-\beta}{2}\cos.\frac{\alpha+\beta}{2} = \tfrac{1}{2}\{\sin.\alpha - \sin.\beta\}$$

$$(15) \quad . \; . \; . \quad \cos.\frac{\alpha+\beta}{2}\cos.\frac{\alpha-\beta}{2} = \tfrac{1}{2}\{\cos.\alpha + \cos.\beta\}$$

$$(16) \quad . \; . \; . \quad \sin.\frac{\alpha+\beta}{2}\sin.\frac{\alpha-\beta}{2} = \tfrac{1}{2}\{\cos.\beta - \cos.\alpha\}$$

A formula derived from these equations will be of material use in the next section. It is obtained by dividing the fourteenth equation by the thirteenth; whence there arises

$$\frac{\cos.\frac{\alpha+\beta}{2}}{\sin.\frac{\alpha+\beta}{2}} \times \frac{\sin.\frac{\alpha-\beta}{2}}{\cos.\frac{\alpha-\beta}{2}} = \frac{\sin.\alpha - \sin.\beta}{\sin.\alpha + \sin.\beta}$$

or, art. 120,

$$\text{cotan.}\frac{\alpha+\beta}{2} \times \tan.\frac{\alpha-\beta}{2} = \frac{\sin.\alpha - \sin.\beta}{\sin.\alpha + \sin.\beta}$$

or, finally, art. 50,

$$(17) \quad . \; . \; . \quad \frac{\tan.\tfrac{1}{2}(\alpha-\beta)}{\tan.\tfrac{1}{2}(\alpha+\beta)} = \frac{\sin.\alpha - \sin.\beta}{\cos.\alpha + \cos.\beta}.$$

125. The terms sine a, cos. a, tan. a, &c. art. 50, have been used to denote the ratios of the sides of a right angled triangle, one of whose angles is a: but whilst treating of these *direct* relations of angles it must frequently happen that we have occasion to mention the *converse* relations.

If, for example, there is given the equation

$$\sin.\, a = b;$$

This expression informs us that a is an angle whose sine is b; but such expressions as—"the angle whose sine is

b"—or—"the angle whose cosine is b," are too long and involved to suit the brevity of mathematical language; and, accordingly, analysts have endeavoured to comprehend each of these expressions in a single term.

The elements of algebra, as they are wholly founded on such abbreviations, will present us with examples similar to the case we are considering.

The equation,

$$a^2 = b$$

offers an instance of this kind: the direct relation is that of a number b equal to the square of a; and the notation employed expresses this relation under an abbreviated form; but the direct relation gives rise to another that is its converse, the relation namely of a to b:—"a is a number whose square is b;" and the object now is to attach a sign to the letter b that shall express in a single term these words—"a number whose square is b." Algebraists, led by analogy, endeavoured to comprehend the relation in question under the notation used for powers; they assumed that when b was the second power of a, the latter was also some unknown power of the former; and putting x for the co-efficient of this power, they had only to solve the equation

$$(b^x)^2 = b.$$

in order to discover, not only whether the assumption was admissible, but also, what number should be substituted for x.

The theory of exponents reduced this equation to

$$b^{2x} = b$$

whence it appeared that

$$2x = 1$$

and, finally,

$$x = \tfrac{1}{2}.$$

A similar process of reasoning will apply to the case which now presents itself to our notice.

Suppose, for example, that in an investigation of the relations of a given number of points in space, or of lines diverging from a given point, we met with the equations

$$\sin. a = b$$
$$\sin. b = c:$$

substituting for b, in the second equation, its value obtained from the first, there arises the expression

$$\sin. \sin. a = c$$

To comprehend the meaning attached to the compound sign on the left hand side, we must consider;

First, that a is a number expressing the relation of some angle to the unit of angles.

Secondly, that a right angled triangle has been constructed, having one of its acute angles equal to the angle denoted by a; and that sin. a—a number—is the ratio of two sides of this triangle.

Having in this way recalled to mind the meaning of the number sin. a, we must select an angle whose ratio to the unit of angles is equal to that number; and, constructing a right angled triangle with one of its acute angles equal to the angle so selected, find in this triangle the ratio of the opposite side to the hypothenuse.

The ratio so found is the number expressed by the compound symbol sin. sin. a.

Continuing the process of reasoning, we should perceive, in like manner, that

$$\sin. \sin. \sin. a$$

was a result derived by eliminating b and c from the equations

$$\sin. a = b$$
$$\sin. b = c$$
$$\sin. c = d;$$

and we should further perceive, that such a result assumes the successive construction of the three right angled triangles to which these equations refer.

The view we have here taken, leads us to expect the occurrence of such quantities as sin. sin. a, sin. sin. sin. a, &c. ; and as they do, in fact, present themselves not unfrequently in analysis, it may be useful to provide abbreviations for them. Algebraists have agreed to use, in this case, the notation employed for powers, and have represented

sin. sin. a - - - - by - - - - $\sin.^2 a$*
sin. sin. sin. a - - by - - - - $\sin.^3 a$
sin. sin. sin. sin. a by - - - - $\sin.^4 a$, &c. &c.

The figure placed above the sine will thus signify the number of right angled triangles successively constructed, or, in the peculiar language of analysis, the number of times which the sine is *taken*.

To discover the rules by which such expressions are combined, we have merely to put for these abbreviated expressions the values that, by convention, they stand for. If we meet, for example, with

$$\sin.^2 \{\sin. a\} ;$$

the symbol $\sin.^2$ has been put, by convention, for sin. sin. a; and here we have,

$$\sin.^2 \{\sin. a\} = \sin. \sin. \{\sin. a\} :$$

but sin. $\{\sin. a\}$ is the same thing as sin. sin. a, since a difference in their signification could only arise from an

* Note *b*.

agreement to that effect, an agreement that we have not made.

The same remark applies to the left hand number of the equation; and, hence,

$$\sin.^2 \{\sin. a\} = \sin.^2 \sin. a = \sin. \sin. \sin. a = \sin.^3 a.$$

Pursuing this method of analysis with other combinations of the same kind, we shall readily demonstrate the exponents to follow a very simple law.

"In any equation of the form

$$\sin.^m \sin.^n \sin.^p a = \sin.^q a$$

the exponent, q, is the sum of the exponents, m, n, and p."

This rule, it will be seen, exactly agrees with that used in algebra for multiplying together different powers of the same quantity: the equation

$$x^m \times x^n \times x^p = x^q$$

equally leading to the result, $q = m + n + p$: and hence we are led to expect the converse notation in the one case, to be derived by a process substantially the same with that used for deducing the converse notation of the other.

According to this view of the subject, a number x can be found, which will enable us to express the words, "the angle whose sine is b," by the symbol "$\sin.^x b$."

To investigate whether such is the case, we must have recourse to the equation

$$\sin. a = b;$$

where a represents the angle implied by the words, "the angle whose sine is b"; or where, if the notation in question can be established,

$$a = \sin.^x b.$$

Substituting for b its value, the former of these two equations becomes,

$$\sin.\ \sin.^{x} b = b,$$

or, which is the same thing,

$$\sin.^{1} \sin.^{x} b = b.$$

And again, this equation, by the law we have established for these exponents, is the same as

$$\sin.^{1+x} b = b,$$

but b, by our notation, must be equivalent to $\sin.^{0} b$, for since the exponent placed over the sine denotes how often it has been "taken," if that exponent is zero, the sine of the angle has not been taken at all, and the symbol must then signify the angle itself.

The last equation then becomes,

$$\sin.^{1+x} b = \sin.^{0} b,$$

whence,

$$1 + x = o$$
$$x = -1.$$

And

$$\sin.^{-1} b$$

will thus be an appropriate symbol to represent the words "an angle whose sine is b."

The equation

$$\sin.\ a = b$$

leads, we have seen, to the expression

$$a = \sin.^{-1} b; \quad . \ . \ . \ . \ . \ 18$$

but if, in the first of these two equations, we had written $\sin.^{-1}$ on both sides, it would have become,

$$\sin.^{-1} \sin.^{1} a = \sin.^{-1} b,$$

or, by the rules already developed,

$$\sin.^{\circ} a = \sin.^{-1} b$$

or,

$$a = \sin.^{-1} b.$$

This result agreeing with the former, it is evident, that if we precede one side of an equation by $\sin.^{-1}$, it will be equal to the other side preceded by $\sin.^{-1}$; a result, indeed, sufficiently obvious; since whatever operation is performed on one side of an equation, it must produce a result agreeing with that which would have been obtained by performing the same operation on the other side.

A number, a, with "sine" placed before it, is said to have its *sine taken*, and a number, a, with "sine^{-1}" placed before it, is said to have its *inverse sine taken*.

The meaning of these expressions will be sufficiently understood from the detailed explanation that has preceded; but, to avoid all error on the subject, we shall remind the student of the operations signified by the symbols in question.

The first symbol, it will be recollected, views a as an angle, and implies, first, that a right angled triangle is to be constructed, having a for one of its acute angles; and, secondly, that we are to measure the *ratios* between the opposite side and the hypothenuse of this triangle.

The operation implied by the second symbol proceeds in an inverse order. The latter supposes the construction of a right angled triangle; but viewing a as the ratio of a side to the hypothenuse, it directs us to measure the *angle* to which the side in question is opposed.

The remarks that have been made with respect to the sine will, of course, apply to the other trigonometrical

ratios, and thus, we shall have cosine ^{-1}a, tan. ^{-1}a, &c. for the respective angles that have cosines or tangents equal to a.

Such a notation will necessarily require rules, by means of which the various combinations that it is susceptible of may be performed; and with some of the most useful of these rules we shall now terminate the subject.

126. The transformations most frequently wanted are those of one inverse ratio into another: we may wish, for example, having the inverse sine of a, to change it into an inverse cosine of the same number.

The process by which such a transformation is to be obtained will be evident;

For, since

$$\text{cosine} = \sqrt{1 - \sin.^2},$$

if a is the sine of an angle, the square root of $1 - a^2$ must be its cosine, or the same angle will be expressed indifferently by

$$\sin.^{-1} a,$$

or,

$$\cos.^{-1} \sqrt{1 - a^2}.$$

The same reasoning applies to the remaining ratios, and deduces the formulæ,

$$\begin{aligned}\sin.^{-1} a &= \cos.^{-1} \sqrt{1 - a^2},\\ &= \tan.^{-1} \frac{a}{\sqrt{1 - a^2}}\\ &= \sec.^{-1} \frac{1}{\sqrt{1 - a^2}} \quad \ldots (19)\\ &= \cot.^{-1} \frac{\sqrt{1 - a^2}}{a}\\ &= \operatorname{cosec.}^{-1} \frac{1}{a}\end{aligned}$$

And in a similar manner we might deduce other relations of a more complicated character, but as the reasoning in the latter case is less easily remembered, it may not be amiss to replace it by an analytical investigation.

Let it be supposed, for example, that we are acquainted with two sines; and, without requiring each of the angles separately, wish to determine their sum. Assuming a and b for the given sines, the proposition requires that we should determine the value of x, in the equation:

$$\sin.^{-1} a + \sin.^{-1} b = \sin.^{-1} x:$$

Now, taking the sine of both sides, this equation becomes,

$$\begin{aligned} x &= \sin. \{\sin.^{-1} a + \sin.^{-1} b\} \\ &= \sin.\sin.^{-1} a \times \cos.\sin.^{-1} b + \sin.\sin.^{-1} b \cos.\sin.^{-1} a \\ &= \sin.\sin.^{-1} a \cos.\cos.^{-1}\sqrt{1-b^2} + \sin.\sin.^{-1} b \cos.\cos.^{-1}\sqrt{1-a^2} \\ &= a\sqrt{1-b^2} + b\sqrt{1-a^2} \end{aligned}$$

or,

$$\sin.^{-1} a + \sin.^{-1} b = \sin.^{-1}\{a\sqrt{1-b^2} + b\sqrt{1-a^2}\} \quad . \, 20$$

And by a similar process we may obtain any of the inverse formulæ that are found in the annexed table.

Most useful relations of three directions that lie in one plane and diverge from a common point.

DIRECT RELATIONS.

$$\mathrm{Sin}.a^2+\cos.a^2=1 \qquad 1+\tan.a^2=\sec.a^2, \qquad 1+\cot.a^2=\mathrm{cosec}.a^2$$

$$\tan.a=\frac{\sin.a}{\cos.a} \qquad \cot.a=\frac{1}{\tan.a}=\frac{\cos.a}{\sin.a} \qquad \sec.a=\frac{1}{\cos.a} \qquad \mathrm{cosec}.a=\frac{1}{\sin.a}$$

$$\sin.(a\pm b)=\sin.a\cos.b\pm\cos.a\sin.b \qquad \cos.(a\pm b)=\cos.a\cos.b\mp\sin.a\sin.b$$

$$\tan.(a\pm b)=\frac{\tan.a\pm\tan.b}{1\mp\tan.a\tan.b}$$

$$\sin.a\cos.b=\tfrac{1}{2}\sin.(a+b)+\tfrac{1}{2}\sin.(a-b) \qquad \cos.a\sin.b=\tfrac{1}{2}\sin.(a+b)-\tfrac{1}{2}\sin.(a-b)$$

$$\cos.a\cos.b=\tfrac{1}{2}\cos.(a+b)+\tfrac{1}{2}\cos.(a-b) \qquad \sin.a\sin.b=\tfrac{1}{2}\cos.(a-b)-\tfrac{1}{2}\cos.(a+b)$$

$$\sin.a+\sin.b=2\sin.\tfrac{1}{2}(a+b)\cos.\tfrac{1}{2}(a-b) \qquad \cos.a+\cos.b=2\cos.\tfrac{1}{2}(a+b)\cos.\tfrac{1}{2}(a-b)$$

$$\sin.a-\sin.b=2\sin.\tfrac{1}{2}(a-b)\cos.\tfrac{1}{2}(a+b) \qquad \cos.a-\cos.b=2\sin.\tfrac{1}{2}(a-b)\sin.\tfrac{1}{2}(a+b)$$

$$\tan.a+\tan.b=\frac{\sin.(a+b)}{\cos.a\cos.b} \qquad \tan.a-\tan.b=\frac{\sin.a-b}{\cos.a\cos.b}$$

$$\sin.1\,a=\sin.a \qquad \cos.a=\cos.a$$

$$\sin.2\,a=2\sin.a\cos.a \qquad \cos.2\,a=2\cos.a^2-1$$

$$\sin.3\,a=3\sin.a-4\sin.a^3 \qquad \cos.3\,a=4\cos.a^3-3\cos.a$$

$$\sin.4\,a=(4\sin a-8\sin.a^3)\cos.a \qquad \cos.4\,a=8\cos.a^4-8\cos.a^2+1$$

. . . .

$$\sin.a=\sin.a \qquad \cos.a=\cos.a$$

$$2\sin.a^2=1-\cos.2\,a \qquad 2\cos.a^2=1+\cos.2\,a$$

$$4\sin.a^3=3\sin.a-\sin.3\,a \qquad 4\cos.a^3=3\cos.a+\cos.3\,a$$

$$8\sin.a^4=3-4\cos.2\,a+\cos.4\,a \qquad 8\cos.a^4=3+4\cos.2\,a+\cos.4\,a$$

. . . .

$$\sin.\tfrac{1}{2}a=\sqrt{\tfrac{1}{2}-\tfrac{1}{2}\cos.a} \qquad \cos.\tfrac{1}{2}a=\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\cos.a.}$$

$$\sin.a^2-\sin.b^2=\cos.b^2-\cos.a^2=\ldots\ldots\sin.(a+b)\sin.(a-b)$$

$$\cos.a^2-\sin.b^2=\ldots\ldots\cos.(a+b)\cos.(a-b)$$

$$\tan.a^2-\tan.b^2=\ldots\ldots\frac{\sin.(a+b)\sin.(a-b)}{\cos.a^2\cos.b^2}$$

$$\cot.a^2-\cot.b^2=\ldots\ldots\frac{-\sin.(a+b)\sin.(a-b)}{\sin.a^2\sin.b^2}$$

$$\frac{\sin.a+\sin.b}{\sin.a-\sin.b}=\frac{\tan.\frac{1}{2}(a+b)}{\tan.\frac{1}{2}(a-b)} \qquad \frac{\sin.(a+b)}{\sin.a+\sin.b}=\frac{\cos.\frac{1}{2}(a+b)}{\cos.\frac{1}{2}(a-b)} \qquad \frac{\sin.(a+b)}{\sin.a-\sin.b}=\frac{\sin.\frac{1}{2}(a+b)}{\sin.\frac{1}{2}(a-b)}$$

Chap. I. Detailed analysis of the relations of direction; and of the relations peculiar to three, to four, and to a greater number of points.

Most useful relations of angles about a point and in one plane.

$$\frac{\sin.a\pm\sin.b}{\cos.a+\cos.b}=\frac{\cos.b-\cos.a}{\sin.a\mp\sin.b}=\tan.\tfrac{1}{2}(a\pm b)$$

$$\frac{\tan.a\pm\tan.b}{\tan.a\mp\tan.b}=\frac{\cot.a\pm\cot.b}{\pm\cot.b-\cot.a}=\frac{\sin.(a\pm b)}{\sin.(a\mp b)}$$

$$\frac{\tan.a+\tan.b}{\cot.a\mp\cot.b}=\tan.a\tan.b$$

$$\frac{\sin.a}{1+\cos.a}=\frac{1-\cos.a}{\sin.a}=\tan.\tfrac{1}{2}a \qquad \frac{\sin.a}{1-\cos.a}=\frac{1+\cos.a}{\sin.a}=\cot.\tfrac{1}{2}a$$

$$\sin.(\tfrac{1}{4}\pm a)=+\cos.a \qquad \cos.(\tfrac{1}{4}\pm a)=\mp\sin.a \qquad \tan.(\tfrac{1}{4}\pm a)=\mp\cot.a$$

$$\sin.(\tfrac{1}{2}\pm a)=\mp\sin.a \qquad \cos.(\tfrac{1}{2}\pm a)=-\cos.a \qquad \tan.(\tfrac{1}{2}\pm a)=\pm\tan.a$$

$$\sin.(\tfrac{3}{4}\pm a)=-\cos.a \qquad \cos.(\tfrac{3}{4}\pm a)=\pm\sin.a \qquad \tan.(\tfrac{3}{4}\pm a)=\mp\cot.a$$

$$\sin.(1\pm a)=\pm\sin.a \qquad \cos.(1\pm a)=+\cos.a \qquad \tan.(1\pm a)=\pm\tan.a$$

INVERSE RELATIONS.

$$\sin.^{-1}a=\cos.^{-1}\sqrt{1-a^2}=\tan.^{-1}\frac{a}{\sqrt{1-a^2}}=\sec.^{-1}\frac{1}{\sqrt{1-a^2}}=\cot.^{-1}\frac{\sqrt{1-a^2}}{a}=\text{cosec.}^{-1}$$

$$\cos.^{-1}a=\sin.^{-1}\sqrt{1-a^2}=\tan.^{-1}\frac{\sqrt{1-a^2}}{a}=\sec.^{-1}\frac{1}{a}=\cot.^{-1}\frac{a}{\sqrt{1-a^2}}=\text{cosec.}^{-1}\frac{1}{\sqrt{1-}}$$

$$\tan.^{-1}a=\sin.^{-1}\frac{a}{\sqrt{1+a^2}}=\cos.^{-1}\frac{1}{\sqrt{1+a^2}}=\sec.^{-1}\sqrt{1+a^2}=\cot.^{-1}\frac{1}{a}=\text{cosec.}^{-1}\frac{\sqrt{1+a}}{a}$$

$$\sin.^{-1}a+\sin.^{-1}b=\sin.^{-1}\left\{a\sqrt{1-b^2}+b\sqrt{1-a^2}\right\}$$

$$\cos.^{-1}a+\cos.^{-1}b=\cos.^{-1}\left\{ab-\sqrt{1-a^2}\sqrt{1-b^2}\right\}$$

$$\tan.^{-1}a+\tan^{-1}b=\tan.^{-1}\left\{\frac{a+b}{1-ab}\right.$$

$$\tan.^{-1}a-\tfrac{1}{8}=\tfrac{1}{2}\tan.^{-1}\tfrac{1}{2}\left(a-\frac{1}{a}\right),\ \tan.^{-1}\frac{a\sqrt{3}}{a+2b}-\tan.^{-1}\sqrt{3}=-\tan.^{-1}\frac{b\sqrt{3}}{2a+b}.$$

Note.—The whole of plane space about a point is denoted by unity.

SECTION II.

RELATIONS OF THREE POINTS, OR PLANE TRIGONOMETRY.

Notation best adapted to the relations of three points —relations expressed in terms of the opposite sides and angles—relations of the three sides—relations of the angles in terms of the sides—any three parts except the three angles will determine the remainder—logarithmic expressions—two sides and included angle—three sides —ambiguity of some of the formulæ—properties common to all plane triangles—case which admits two solutions—properties of particular triangles—relations that the parts of triangles have with lines drawn in and about the latter.

127. The relative positions of three points involve, we have seen, three distances and three angles, whose relations are contained in the equations (1), art. 51.

The resolution of these equations, and the development of the relations they contain, is a problem that belongs to algebra; and, by the rules of that science, without any further reference to geometry, we could

now discover all the combinations of which the relations of three points are susceptible.

But the process of elimination, one of the most tedious in algebra, is facilitated by employing, in conjunction with the equations (1), others deduced by a different geometrical analysis; and wherever this mixed process enables us to avoid prolixity, we shall not scruple to avail ourselves of it.

The notation we have hitherto used may, also, in the present instance, be laid aside with advantage, both as more comprehensive than the case requires, and as yielding, in simplicity, to a notation peculiarly adapted to the relations of three points.

NOTATION EMPLOYED.

According to this last method of naming the parts of a triangle, the angles are denoted by any of the roman capitals, whilst italic letters of the same name are put for the opposite sides.

128. Analysing the given triangle into two right angled triangles, and denoting the perpendicular by p, we have,

$$\frac{p}{c} = \text{sin. A}$$

$$\frac{p}{a} = \text{sin. C};$$

and dividing the first of these equations by the second,

$$\frac{a}{c} = \frac{\text{sin. A}}{\text{sin. C}} : \quad . \quad . \quad . \, \alpha$$

similar expressions for the ratio of any two of the sides may be obtained by the same process, but the principle

of symmetry will render such additional investigations unnecessary, and enable us to deduce from the equation α the three following equations:

$$\frac{a}{c} = \frac{\sin. A}{\sin. C}$$

$$\frac{b}{c} = \frac{\sin. B}{\sin. C} \quad \ldots\ldots \quad 21$$

$$\frac{a}{b} = \frac{\sin. A}{\sin. B}$$

which assert the ratio of any two sides to be the same with the ratio of the sines of the opposite angles.

129. Multiplying the first of the equations,

$$a = b \cos. C + c \cos. B$$

$$b = a \cos. C + c \cos. A$$

$$c = a \cos. B + b \cos. A$$

by a, the second by $-b$, and the third by $-c$; and, adding the equations together, there results,

$$a^2 = b^2 + c^2 - 2\,bc \cos. A$$

or,

$$a = \sqrt{\{b^2 + c^2 - 2\,bc \cos. A\}} \quad \ldots \quad 22$$

From which formulæ, by interchanging the letters, we derive similar expressions for the other sides.

This value of a will be the greatest, when $A = \frac{1}{2}$; or when the point A lies between B and C, and the three points are in one straight line.

The formula then becomes

$$a = b + c:$$

but for any other value of A,

$$a < b + c.$$

And as similar results are true for b and c, we con-

Chap. I. Detailed analysis of the relations of direction; and of the relations peculiar to three, to four, and to a greater number of points.

Art. 129. Relations of the three sides. Art. 130. Relations of the angles in terms of the sides. Art. 131. Any three relations except the three angles will determine the remainder.

clude, that, when the points do not lie in the same direction, the three quantities, a, b and c, are limited by the condition that any two of them shall be greater than the third.

130. The equation 22 also gives,

$$A = \cos.^{-1}\left\{\frac{b^2 + c^2 - a^2}{2\,bc}\right\} \quad . \; . \; . \quad 23$$

from which, by interchanging the letters, expressions for the other angles may be derived.

This last equation may be put under the form,

$$A = \cos.^{-1}\left\{\frac{(b + c - a)\,(b + c + a)}{2\,bc} - 1\right\}$$

where we observe, that, when a is greater than $b + c$, the cosine of A is greater than unity; an impossible result, that proves we have assumed, among the distances of three points, relations that could not exist.

131. The equations 1, art. 51, which involve the whole of plane trigonometry, containing six quantities, it is necessary that three of the latter should be known, in order to determine the remainder: in the particular case, however, where the three given things are three angles, the data will not be sufficient; for, as all similar triangles have their corresponding angles equal, it must follow that, from a knowledge of the angles, we can merely determine the form, and not the magnitude of the triangle. This result is also manifested in the algebraic solution of the problem, the value of a side when expressed in terms of the three angles assuming the indeterminate form $\frac{0}{0}$.

132. In every other case, where three of the six parts are given, the formulæ 21, 22 and 23 will enable us to determine the parts which are sought: but although these formulæ present to the eye a complete solution of every case that can occur, they are not, in their present form, adapted to practice.

The objection applies, indeed, only to the expressions 22 and 23, which do not readily permit the use of logarithms.

The processes of multiplication and division being reduced by that method of calculation to the more simple processes of addition and subtraction, all expressions to which logarithms are to be applied, should be previously reduced, if possible, to factors; and, to factors whose component parts are connected by the signs of addition and subtraction.

As the formula 22, that expresses a side in terms of the other two sides and the angle included between them, does not admit of such a transformation, this case has been resolved by a different process, which we shall now explain.

Assuming a, b and c for the given parts, we have, from the equations 21,

$$\frac{a}{c} = \frac{\sin. A}{\sin. C}, \quad \frac{b}{c} = \frac{\sin. B}{\sin. C};$$

and, adding and subtracting these equations, there arises,

$$\frac{a+b}{c} = \frac{\sin. A + \sin. B}{\sin. C}$$

$$\frac{a-b}{c} = \frac{\sin. A - \sin. B}{\sin. C}$$

whence, by division,

$$\frac{a-b}{a+b} = \frac{\sin. A - \sin. B}{\sin. A + \sin. B}.$$

But, in article 124, equation 17, we have a value of the right hand member of this equation, that changes the latter into

$$\frac{a - b}{a + b} = \frac{\tan. \frac{1}{2}(A - B)}{\tan. \frac{1}{2}(A + B)} \quad . \; . \; . \; . \; 24$$

And since

$$A + B = \tfrac{1}{2} - C \quad . \; . \; . \; . \; 25$$

we shall know three out of the four terms that enter into the equations 24.

Having determined from this last the value of $A - B$, we have, then,

$$\tfrac{1}{2}(A + B) + \tfrac{1}{2}(A - B) = A$$
$$\tfrac{1}{2}(A + B) - \tfrac{1}{2}(A - B) = B,$$

And as all the angles, as well as the two given sides, are now known, the side c can be found from the equations 21.

The formula 24 is usually expressed as the following rule:

The ratio between the sum and difference of any two sides, is equal to the ratio between the tangents of half the sum and half the difference of the opposite angles.

133. The expression for an angle in terms of the three sides, admits the transformation to which we have alluded.

For transposing the term unity in the equation, art. 130,

$$\cos. A = \frac{(b + c - a)(b + c + a)}{2\,bc} - 1$$

it becomes,

$$1 + \cos. A = \frac{(b + c - a)(b + c + a)}{2\,bc}$$

The right hand number is expressed in factors of the kind required, and with regard to the left hand number, we observe, art. 119, γ, that it is equivalent to 2 cos. $\frac{1}{2}$ A^2; whence,

$$\cos. \tfrac{1}{2} A = \sqrt{\left\{\frac{(b+c-a)\ (b+c+a)}{4\ bc}\right\}} \quad . \ . \ 26$$

This expression is well adapted to calculation, but it admits of a form still more commodious.

To obtain this last, it is necessary to render the compound factors, of the expression 26, symmetrical, an object effected with most facility, by commencing the process of reduction anew.

The equation 23 gives,

$$\cos. A = \frac{b^2 + c^2 - a^2}{2\ bc};$$

which is the expression that is to be reduced into factors.

But among the compound expressions of the second degree that can be readily resolved into factors, one, the difference of two squares, is remarkable for its simplicity.

Now the numerator of the fraction we are considering, can readily be reduced to the difference of two squares. It is only necessary, for this purpose, to add or subtract the denominator, and the quantity in question will become $(b \pm c)^2 - a^2$.

If the denominator is added, the factors will not be symmetrical, but they assume a symmetrical form when, on the contrary, the denominator is subtracted.

To effect the latter operation, subtract $\frac{2\,bc}{2\,bc}$ on one side, and its value, unity, on the other. The equation becomes,

$$\cos. A - 1 = \frac{(b-c)^2 - a^2}{2\ bc}.$$

Again, art. 101, $1 - \cos. A = 2 \sin. \frac{1}{2} A^2$, and hence,

$$\sin. \tfrac{1}{2} A^2 = \frac{a^2 - (b - c)^2}{4\,bc}$$

resolving the right hand number into its factors,

$$\sin. \tfrac{1}{2} A^2 = \frac{\frac{1}{2}(a + b - c)\,\frac{1}{2}(a + c - b)}{bc}.$$

But if the sum of the three sides, a, b and c, is denoted by s, we have

$$\tfrac{1}{2}(a + b - c) = \tfrac{1}{2} s - c,$$

and

$$\tfrac{1}{2}(a + c - b) = \tfrac{1}{2} s - b;$$

whence, finally, the expression for A becomes

$$\sin. \tfrac{1}{2} A = \sqrt{\left\{\frac{(\frac{1}{2} s - b)(\frac{1}{2} s - c)}{bc}\right\}} \quad . \; . \; . \; 27$$

134. When the quantity sought is an angle, each of the equations 21 gives two solutions. To be convinced of this fact, it is only necessary to reflect, that as the sine of an angle is the same as the sine of its supplement, we cannot determine, from the value of the sine, whether the angle or its supplement is to be taken. The same remark, however, does not apply to the formulæ 24 and 27; for, as an angle is always considered, in these investigations, as less than 180°, the half of any angle must be less than 90°, and the angle found in the tables is, consequently, that which resolves the problem.

The formulæ 24 and 27 may, therefore, be considered as completely resolving the question with reference to which they were deduced; whilst the equations 21 re-

quire, in a particular instance, that we should unite with them results obtained from other considerations.

The properties that we have detailed below, as common to all triangles, are results of this kind; and will enable us to remove the ambiguity in question.

PROPERTIES COMMON TO ALL PLANE TRIANGLES.

135. 1. In every triangle, the sum of the three angles is equal to two right angles, art.

2. The sum of any two sides of a plane triangle is greater than the third side, art.

3. The difference of any two sides of a plane triangle is less than the third side.

Let a, b and c represent the sides, arranged in the order of their magnitude; we have

$$b + c > a$$
$$b = b$$

whence, by subtraction,

$$c > a - b.$$

4. Any side of a plane triangle is less than $\frac{1}{2}$, and greater than $\frac{1}{3}$, the sum of the sides.

Taking the same notation as before,

$$a < b + c$$
$$\therefore 2a < a + b + c$$

or,

$$a < \frac{a + b + c}{2}$$

Again, since the sides are arranged in the order of their magnitude,

$$a > b - c$$
$$2a > 2c$$
$$a = a,$$

And adding these equations, and dividing by 4,

$$a > \frac{a + b + c}{4}$$

5. The greater angle of a plane triangle is opposite to the greater side, and the converse,

For since

$$\frac{\sin. A}{\sin. B} = \frac{a}{b},$$

if a is greater than b, sin. A must be greater than sin. B, and A than B. The latter conclusion, when A and B are each less than $\frac{1}{4}$, follows from art. 50—3, and when either of these angles exceeds $\frac{1}{4}$, the same result may be established by a very simple process.

For since

$$B < \tfrac{1}{2} - A$$

it will follow, first, that if A is greater than $\frac{1}{4}$, B is less than $\frac{1}{4}$, and A, consequently, greater than B; secondly, that if B is greater than $\frac{1}{4}$, it will be yet less than the supplement of A, and sin. B > sin. ($\frac{1}{2}$ — A), or greater than sin. A: and as this result is contrary to the hypothesis, we conclude that A must be greater than B.

And from the same reasoning we may also establish the converse proposition: for since it has been proved that A and B, considered as angles of a triangle, increase and decrease with their sines, the equation

$$\frac{\sin. A}{\sin. B} = \frac{a}{b}$$

will require, that when A is greater, equal, or less than

B, the same relation should be found in the numerator and denominator of the right hand number.

6. A plane triangle can only have one right, or one obtuse angle.

The application of these properties to the problem with a view to which they were introduced, will scarcely require a comment: the difficulty, it will be recollected, art. 134, regarded the choice between an angle and its supplement, and it is now obvious that in selecting from the two angles which the formula presents, we must choose that which is consistent with the properties above demonstrated.

136. Only a single case however occurs, we remarked, when this trial is necessary.

It is that wherein the given things are two sides and an angle opposite to the greater of them.

When the data are two sides and an angle opposite to the less, the formulæ 21 equally apply, but the double values which they give are both solutions.

That such is the case will appear from considering, that, if in any triangle, an angle B is greater than A, the supplement of B, art. 135—5, will be also greater than A; and, consequently, that it is only when the smaller angle is to be found that the criterion we have established applies.

PROPERTIES OF PARTICULAR TRIANGLES.

137. 1. Plane triangles that are equilateral are equiangular and the converse.

These properties follow immediately from the equations

$$\frac{\sin. A}{\sin. B}=\frac{a}{b},\ \frac{\sin. A}{\sin. C}=\frac{a}{c},\ \&c.$$

combined with the property that no angle of a triangle can be so great as the supplement of another of its angles.

137. 2. Isosceles triangles have the angles opposite the equal sides equal, and the converse.

These properties are demonstrated as the preceding.

RELATIONS THAT THE PARTS OF TRIANGLES HAVE WITH LINES DRAWN IN OR ABOUT THE LATTER.

138. 1. In any isosceles plane triangle, a perpendicular drawn from the vertex, will bisect both the base and the vertical angle.

The figure is divided into two right angled triangles, which have a corresponding side and angle equal in each; whence all the remaining parts are equal.

138. 2. In any plane triangle, if we draw a line from the vertex to the middle of the base, the sums of the squares of the other two sides will be equal to twice the sum of the squares of the bisecting line and half base.

Let the vertices of the triangle be ABC, and the middle of the base D: the figure is divided into two triangles by the bisecting line; whence, using the notation of page 109,

$$a^2 = a''^2 + b'^2 - 2\,a''b' \cos.\ a''b'$$
$$b^2 = c^2 + b'^2 - 2\,cb' \quad \cos.\ cb'.$$

But a'' and c are each equal to $\frac{1}{2}a'$, and cos. $cb' =$ — cos. $a''b'$; whence, adding the equations,

$$a^2 + b^2 = 2\{(\tfrac{1}{2}a')^2 + b'^2\}.$$

138. 3. In any plane triangle if a line is drawn bisecting the vertical angle, the segments of the base will have the same ratio as the opposite sides.

Using the same notation, we have, from the two triangles into which the figure is divided,

$$\frac{a}{a''} = \frac{\sin.\ a''b'}{\sin.\ ab'}$$

$$\frac{b}{c} = \frac{\sin.\ cb'}{\sin.\ bb'}$$

But

$$(a''b') = \tfrac{1}{2} - (cb')$$

and

$$(ab') = bb';$$

whence it appears that the right hand members of the two first equations are equal, and there results

$$\frac{a}{a''} = \frac{b}{c}.$$

or

$$\frac{a}{b} = \frac{a''}{c}.$$

APPENDIX TO SECTION II.

EXAMPLES.

1. In a right angled triangle ABC, right angled at C, the following parts are given

$$A = 39^\circ\ 17'$$
$$c = 509$$

required the remaining parts.

To find a.

$$\frac{a}{c} = \sin. A$$
$$a = c \sin. A.$$

$$\begin{array}{lcl} \log.\ 509 & = & 2.7067178 \\ \log.\ \sin.\ \ 39^\circ\ 17' & = & \underline{9.8015106} \\ \text{Sum, rejecting 10,} & = & 2.5082284 = \log.\ 322.277 \end{array}$$

$$a = 322.277.$$

To find b.

$$\frac{b}{c} = \cos. A$$
$$b = c \cos. A$$

$$\begin{array}{lcl} \log.\ 509 & = & 2.7067178 \\ \log.\ \cos.\ 39^\circ\ 17' & = & \underline{9.8887547} \\ \text{Sum, rejecting 10,} & = & 2.5954725 = \log.\ 393.978 \end{array}$$

$$b = 393.978.$$

To find B.

$$\begin{array}{rrr} & 90 & 00 \\ & 39^{\circ} & 17 \\ \hline B = & 50 & 43 \end{array}$$

2. In a right angled triangle ABC, right angled at C, the following parts are given,

$$a = 105.4$$
$$b = 734.6$$

required the parts remaining.

To find A.

$$\frac{a}{b} = \text{tan. } A$$

$$\begin{array}{rl} \text{log. } 105.4 = & 2.0228406 \\ \text{log. } 734.6 = & 2.8660509 \\ \hline \text{Diff. adding 10,} = & 9.1567897 = \text{log. tan. } 8^{\circ}\ 9'\ 54'' \end{array}$$

$$A = 8^{\circ}\ 9'\ 54''$$

To find B.

$$\begin{array}{rrrr} & 180 & 0 & 0 \\ & 8 & 9 & 54'' \\ \hline B = & 171 & 50 & 6 \end{array}$$

To find c.

$$\frac{a}{c} = \text{sin. } A$$

$$c = \frac{a}{\text{sin. } A}$$

$$\begin{array}{rl} \text{log. } 105.4 = & 2.0228406 \\ \text{log. sin. } 8^{\circ}\ 9'\ 54'' = & 9.1523625 \\ \hline \text{Diff., adding 10,} = & 2.8704771 = \text{log. } 742.50 \end{array}$$

$$c = 742.50.$$

Appendix to Section II.

Examples.

3. Given in the plane triangle ABC, the following parts

$$a = 354$$
$$b = 248$$
$$B = 41^\circ\ 36'$$

to find the parts remaining.

To find A.

$$\frac{a}{b} = \frac{\text{sin. A}}{\text{sin. B}}$$

$$\text{sin. A} = \frac{a \text{ sin. B}}{b}$$

$$\text{log. sin. A} = \text{log. } a + \text{log. sin. B} + \text{ar. co. log. } b$$

$$\begin{aligned} \text{log. } 354 &= 2.5490033 \\ \text{log. sin. } 41^\circ\ 36' &= 9.8221198 \\ \text{ar. co. log. } 248 &= \bar{3}.6055483 \\ \text{sum} &= 9.9766714 \end{aligned}$$

This sum is the sine of either $71^\circ\ 23'$, or $108^\circ\ 37'$, and either solution will be consistent with the data.

To find C.

B =	41°	36′		41°	36′	
A =	71	23	or	108	37	
	112	59		150	13	
	180	00		180	00	
Diff.	67	1	or	29	47	. . . C.

To find c.

$$\frac{c}{b} = \frac{\text{sin. C}}{\text{sin. B}}$$

$$\text{log. } c = \text{log. sin. C} + \text{log. } b + \text{ar. co. log. sin. B}$$

Appendix to Section II.

Examples.

log.sin. 67° 1	= 9.9640797	. . 29° 47′	= 9.6961130
log. . . 248	= 2.3944517	. . . .	= 2.3944517
ar.co.log.sin. 41 36	= $\overline{10}$.1778802	. . . .	$\overline{10}$.1778802
sum	= 2.5364116	. . . .	2.2684449

These sums correspond to the logarithms of 343.88 and 185.54; whence

$$c = 343.88$$
or
$$185.54.$$

4. In the plane triangle ABC the following parts are given,

$$a = 4324$$
$$b = 6780$$
$$C = 105^\circ\ 42'$$

it is required from these data to determine the other parts of the triangle.

To find A and B.

$$\frac{b - a}{b + a} = \frac{\tan. \frac{1}{2} \{B - A\}}{\tan. \frac{1}{2} \{B + A\}}$$

$$\text{Log.tan.} \tfrac{1}{2} \{B - A\} = \text{log.tan.} \tfrac{1}{2} \{B + A\} + \text{log.} (b - a) + \text{ar. co. log.} (b + a)$$

6780		180	
4324		105 42	
11104	$b + a$	2)74 18	
2456	$b - a$	32 9	$\frac{1}{2}$ (B + A)

Appendix to Section II.

Examples.

$$
\begin{array}{lrl}
\text{log. tan. } & 32^\circ\ 9' & = 9.7983160 \\
\text{log.} & 2456 & = 3.3902284 \\
\text{ar. co. log.} & 11104 & = \bar{5}.9545205 \\
\text{sum} = \text{log. tan. } & 7^\circ\ 54'\ 51'' & = 9.1430649
\end{array}
$$

$$
\begin{array}{rrrl}
& & 2 & \quad \ldots .22689 \text{ next less.} \\
B - A = & 15 & 49\ 42 & \quad \ldots .31959 \text{ next greater.} \\
B + A = & 74 & 18\ 00 & \quad 7960 \text{ 1st diff.} \\
2)\overline{90} & 7 & 42 & \quad 9270 \text{ 2d diff.} \\
B = & 45 & 3\ 51 & \quad \frac{796}{927} = \frac{51}{60} \\
2)\overline{58} & 28 & 18 & \\
A = & 29 & 14\ \ 9 &
\end{array}
$$

Proof.

$$
\begin{array}{rrrr}
A = & 29 & 14 & 9 \\
B = & 45 & 3 & 51 \\
C = & 105 & 42 & 00 \\
\hline
& 180 & 00 & 00
\end{array}
$$

To find c.

$$\frac{c}{a} = \frac{\text{sin. C}}{\text{sin. A}}$$

$$\text{log. } c = \text{log. } a + \text{log. sin. C} + \text{ar. co. log. sin. A}$$

$$
\begin{array}{lll}
\text{log. sin. } 74^\circ\ 18 = & 9.9834872 & \\
\text{log. } \qquad 4324 = & 3.6358857 & \\
\text{ar.co.log.sin.} 29^\circ 14' 9'' = & \bar{10}.3112195 & \ldots 3112533 \text{ next less} \\
\text{sum.} = \text{log. } 8523 = & 3.9305924 & \ldots .0277 \text{ next g'r.} \\
& & \quad 2256 \text{ diff.} \\
c = 8523 & & \quad \frac{9}{60} = \frac{338}{2256}
\end{array}
$$

5. In a plane triangle ABC the three sides are,

$$a = 4,\ b = 9,\ c = 10;$$

required the three angles.

Appendix to Section II.

Examples.

To find A.

$$\sin.\ \tfrac{1}{2}A = \sqrt{\left\{\frac{(\frac{1}{2}s-b)(\frac{1}{2}s-c)}{bc}\right\}}$$

$$\log.\sin.\ \tfrac{1}{2}A = \tfrac{1}{2}\{\log.(\tfrac{1}{2}s-b) + \log.(\tfrac{1}{2}s-c) + \\ + \text{ar. co. log. } b + \text{ar. co. log. } c\}$$

$a = 4$ — log. $2.5 = 0.3979400$

$b = 9$ — log. $1.5 = 0.1760913$

$c = 10$ — ar. co. log. $9 = \bar{1}.0457575$

$2)\overline{23}$ — ar. co. log. $10 = \bar{2}.9999999$

$11.5 - \frac{1}{2}s$ — $2)\overline{\bar{2}.6197887}$

$2.5 - \frac{1}{2}s - b$ — $\frac{1}{2}$ sum $= \bar{1}.3098943$

$1.5 - \frac{1}{2}s - c$ — 10

log. sin. $11^\circ\ 46'\ 41.6'' = 9.3098943$

2 — 94737 next less.

$A = 23\ 33\ 23.3$ — 100798 next g'r.

$$\begin{array}{l} \text{1st diff. } 4206 \\ \text{2d diff. } 6061 \end{array} = \frac{41.6}{60}$$

To find B.

$$\frac{\sin.\ B}{\sin.\ A} = \frac{b}{a}$$

log. sin. B = log. sin. A + log. b + ar. co. log. a

log.sin.$23^\circ33'23''.3 = 9.6016828$ — 6015703 next less.

log. $9 = 0.9542425$ — 6018600 next g'r.

2897 diff.

ar. co. log. $4 = \bar{1}.3979400$ — $\frac{23.3}{60} = \frac{1125}{2897}$

sum=lo.si.$64^\circ3'20'' = 9.9538653$

448 next less.

9063 next greater.

$$\begin{array}{l} \text{1st diff. } 205 \\ \text{2d diff. } 615 \end{array} = \frac{20}{60}$$

SECTION III.

RELATIONS OF THREE DIVERGENT LINES.

Equations of condition fulfilled by three lines that lie in one plane—relations of the angles formed by three divergent lines—inclinations of the planes included in the relations of three divergent lines—when one of these inclinations is a right angle—without the restriction of the last article—notation proper to three divergent lines—opposite solid angles—use of opposite solid angles in analysis—table of the formulæ used when one solid angle is 90°—apply to the case wherein a plane angle is right—Napier's rules—three parts, in the most general case, determine the remainder—enumeration of data—two plane angles and an opposite solid angle—or the converse—two plane angles and the included solid angle—two solid angles and the interjacent plane angle—another solution of the two preceding cases—three sides—three angles—properties common to every case of three divergent lines—particular relations of three divergent lines—ambiguous cases—appendix.

139. The three lines whose relations are investigated in Sect. I. were assumed to lie in a common plane, and

as in the next article we shall prove this condition essential to the equations of the section alluded to, the latter may be regarded as equations of condition fulfilled by three lines that lie in one plane.

Other equations, however, that are deserving of notice, express the same condition.

To investigate them, draw lines parallel to the given directions, and forming a closed figure, a condition always possible when the directions lie in one plane.

The third form of theorem 9, art. 112, employed according to the rules of art. 113, but with some alteration in the arrangement, will give,

$$\begin{aligned} 0 &= m \cos. mm + m' \cos. m m' + m'' \cos. m m'' \\ 0 &= m \cos. mm' + m' \cos. m'm' + m'' \cos. m'm'' \quad \ldots 28 \\ 0 &= m \cos. mm'' + m' \cos. m'm'' + m'' \cos. m''m'', \end{aligned}$$

Which, regarding the sides m, m' and m'' as the unknown quantities, may be written more simply,

$$\begin{aligned} 0 &= \alpha' \, m + \beta' \, m' + \gamma' \, m'' \\ 0 &= \alpha'' m + \beta'' m' + \gamma'' \, m'' \\ 0 &= \alpha''' m + \beta''' m' + \gamma''' m'' \end{aligned}$$

Applying the method of elimination of Bezout, a method given in most elementary works upon algebra, the values of m, m' and m'', will be expressed by zero divided by a constant factor D : and, consequently, either the sides of the closed figure must all be zero, or we must have the equation of condition D=0.

The first result is inconsistent with the construction; which, far from producing a figure whose sides are of infinitely small magnitude, leaves the magnitude of the sides altogether arbitrary.

The equation D=0 is, therefore, established; and, in fact, is consistent with our last remark respecting the

magnitude of the sides; since the symbol $\frac{o}{o}$ which then appears for the values of m, m' and m'', implies, among its other significations, an indeterminate quantity.

Developing the value of D by Bezout's rule, the equation

$$D=0$$

becomes

$$\alpha'\beta''\gamma''' - \alpha'\gamma''\beta''' + \gamma'\alpha''\beta''' - \beta'\alpha''\gamma''' + \beta'\gamma''\alpha''' - \gamma'\beta''\alpha''' = 0,$$

which, substituting for the several letters their values, assumes the form

$$(29)\ldots 1 - \cos. mm'^2 - \cos. mm''^2 - \cos. m'm''^2 + 2 \cos. mm' \cos. mm'' \cos. m'm'' = 0.$$

The angles in this equation of condition are those which measure the mutual inclination of the sides; but we may write it under a form that applies to any three angles whose sum is $\frac{1}{2}$.

For this purpose we must recal some of the results obtained in the preceding investigations.

In the method of estimating angles assumed in the first formula of art. 112, we showed the direction of the last side to be reversed.

The angles so measured agreed with the interior angles of the figure.

The interior angles of a figure of three sides are equal to $\frac{1}{2}$.

Hence, changing the sign of the last term in equation 29, to allow for the change of direction above mentioned; and taking a, b and c to represent any three angles whose sum is $\frac{1}{2}$, the equation may be written,

$$(30)\ldots 1 = \cos. a^2+\cos. b^2+\cos. c^2+2 \cos. a \cos. b \cos. c.$$

And substituting in this last $\frac{1}{4}-a$ for a, $\frac{1}{4}-b$ for b, $\frac{1}{4}-c$ for c, it takes the form

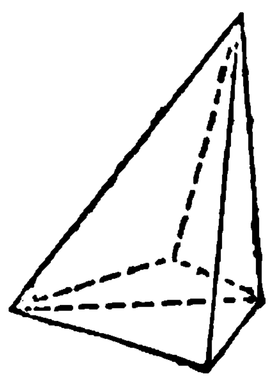
$$(31)\ldots 1=\sin. a^2+ \sin. b^2+ \sin. c^2+2 \sin. a \sin. b \sin. c;$$

which expresses the condition that $a+b+c=\frac{1}{4}$.

The equations 29, 30 and 31, less frequently used than the equations of Sect. I. are yet important; CARNOT has founded on them his analysis of polygons and polyedrons.

140. If the three angles are not in one plane, they do not admit of an equation between them; since a knowledge of two, art. 85, does not suffice to determine the third. They admit, however, of an imperfect relation, which, as in the corresponding case of three lines that form a closed figure, limits any one to be less than the sum of the other two: a condition, we recollect, that assumes numerous forms, art. 135—2, 3, 4.

Fig. 198.



A

For applying two rectangular pyramids to the solid angle which is formed by the three plane angles in question, the bases of the pyramids will form one of the plane angles.

But, art. 61, in each pyramid, confining ourselves to the angles at A, the acute angle of the base is less than the acute angle of the oblique face.

Hence the angle made up by the sums of the bases is less than the sum of the other two angles.

141. But although no definite relation exists between

the plane angles which are formed by three lines diverging from a point; the directions of those lines do still involve relations; since the plane angles are connected with the inclinations of the planes that contain them, and with the solid angle they include.

If the lines diverge from a point A, they may be considered as tending towards points B, C and D, taken at pleasure in their directions; and may therefore be denoted, art. 39, by the letters a, a' and a''.

The symbols for the planes that pass through a and a', a and a'', and a' and a'', will, by the same article, be A, $A_{,}$, A′.

142. In the first case that we shall consider, one of three angles is a right angle; and as a rectangular pyramid can then be found, that shall exactly fill the solid angle at A, the relations are reduced to those of the types of solid figures

The properties of that type give, art. 61,

$$\cos. aa'' = \cos. aa' \cos. a'a'' \quad \ldots\ldots \quad 32.$$

$$\sin. aa' = \sin. aa'' \sin. A_{,} A' \quad \ldots\ldots \quad 33.$$

To which may be added,

$$\cos. A_{,} A' = \sin. A_{,} A \cos. a\,a' \quad \ldots\ldots \quad 34.$$

an equation that we shall derive in art. 147.

Referring again to the type, and supposing BCD situated at its angle, we have

$$\frac{c}{a'} = \sin. a'a''$$

$$\frac{c}{b} = \cot. A_{,} A'$$

$$\frac{b}{a'} = \tan. a\,a'$$

And multiplying together the two last equations, and comparing their product with the first, we obtain,

$$\sin. a'a'' = \tan. a'a \cot. A_{,} A' \quad . \ . \ . \ . \ . \ 35.$$

By analogy, we have,

$$\sin. aa' = \tan. a'a'' \cot. A_{,} A,$$

multiplying which into the equation 35, and recollecting that $\tan. = \frac{\sin.}{\cos.}$, there arises,

$$\cos. aa' \cos. a'a'' = \cot. A_{,} A' \cot. A_{,} A;$$

or, from equation 32,

$$\cos. aa'' = \cot. A_{,} A' \cot. A_{,} A \quad . \ . \ . \ . \ 36.$$

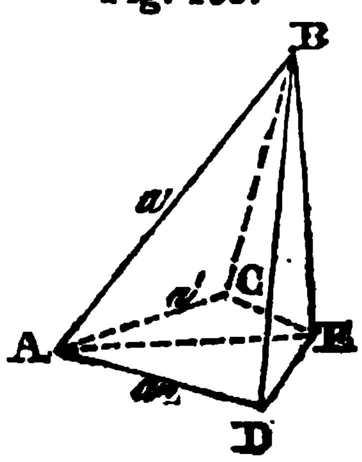
To this may be added,

$$\cos. A_{,} A' = \cot. aa'' \tan. a'a''; \quad . \ . \ . \ . \ 37.$$

an equation readily derived from what has been here done, but which we shall defer investigating until the results of art. 143 and 145 have been established.

143. Passing from the relations of three divergent lines restricted by the condition of the preceding article to those of three divergent lines that are unrestricted by any condition; we proceed to analyse the latter case by the use of two rectangular pyramids.

Fig. 199.



Retaining the notation of art. 141, and analysing the relations sought by means of the closed figure ACEDA, fig. 198, we have, art. 112,

$$a' = a'' \cos. a''a' + d \cos. d\,a';$$

and from an analysis by triangles

$$a' = a \cos. a\,a'$$
$$a'' = a \cos. a\,a''$$
$$d = b' \cos. A_{,} A'$$
$$b' = a \sin. a\,a''.$$

Whence, by substitution,

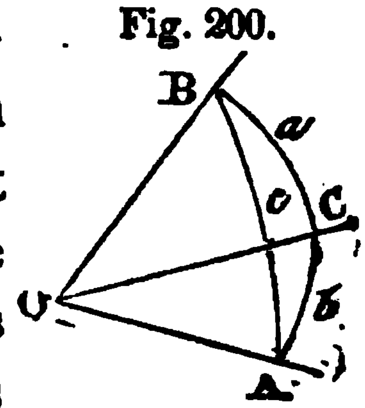
$$\cos. aa' = \cos. aa'' \cos. a''a' + \sin. aa'' \cos. d\,a' \cos. A_{,}A'.$$

But $(da') = \frac{\pi}{2} - (a'a'')$, and $\cos. da' = \sin. a'a''$; whence, inverting the order of the terms, the formula for $\cos. aa'$ becomes,

$$\cos. aa' = \sin. aa'' \sin. a'a'' \cos. A_{,}A' + \cos. aa'' \cos. a''a' \quad . \; . \; 38.$$

144. When the number of divergent lines does not exceed three, the notation we have generally adopted is less convenient than a notation similar to that used in art. 128. According to the latter method of expressing the relations of such lines, the angles formed by the divergent lines are named a, b and c; whilst the capitals A, B and C, are used to express the solid angles formed by the planes which pass through the divergent lines.

Fig. 200.



The letter A denotes the solid angle opposite to a, or the solid angle formed by the inclination of the planes in which a does not lie; and a similar remark applies to the other letters.

The equation 38, expressed by this notation, will be

$$\cos. a = \sin. b \sin. c \cos. A + \cos. b \cos. c.$$

And as the reasoning applied to the solid angle A, and the parts related to it, applies equally to either of the other angles and their corresponding parts, we must have, by analogy, the equations

$$\left.\begin{aligned} \cos. a &= \sin. b \sin. c \cos. A + \cos. b \cos. c \\ \cos. b &= \sin. a \sin. c \cos. B + \cos. a \cos. c \\ \cos. c &= \sin. a \sin. b \cos. C + \cos. a \cos. b \end{aligned}\right\} \; . \; . \; . \; . \; . \; 39.$$

which include all the relations of three divergent lines; and correspond with the equations 1, art. 51, deduced for three lines that form a closed figure.

145. Some, however, of the relations implied in the equations 39 cannot without a tedious elimination be developed from them; whilst they admit a far more commodious investigation by means of a solid angle that has remarkable relations with the angle we are considering.

The latter, we recollect, is the solid angle formed at O, fig. 200, by the plane angles a, b, c, that are contained in planes whose inclinations have been denoted by capital letters of the same name.

Now retaining a similar notation for the new solid, but distinguishing by accents the letters referring to the latter from those referring to the solid angle at O; the former solid is that wherein

$$a' = \tfrac{1}{2} - A,\ b' = \tfrac{1}{2} - B,\ c' = \tfrac{1}{2} - C \quad . \; . \; . \; . \quad 40.$$

The demonstration in art. 85 and 40, proving that any three plane angles, two of which are greater than the third, are capable of containing a solid angle, sufficiently establishes the possibility of finding a solid whose parts shall fulfil the equations 40; and when these are fulfilled we shall prove that

$$A' = \tfrac{1}{2} - a,\ B' = \tfrac{1}{2} - b,\ C' = \tfrac{1}{2} - c \quad . \; . \; . \; . \quad 41.$$

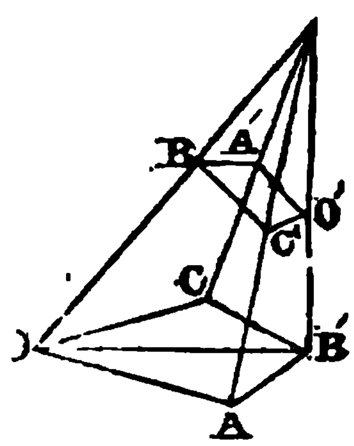
The solid angle connected with that at O by these simple relations has been called the supplemental, or the polar angle of the latter; but these terms, which have their origin in different views of the subject, are little adapted to the method of investigation that we have followed; the latter term would here be altogether without

meaning, and the former is deficient in accuracy; the parts of the two solids are indeed supplementary, but not so the solids themselves, which added together do not produce a constant sum.

The term *opposite solid angles* seems least objectionable, and, although not free from ambiguity, will perhaps sufficiently indicate the angles whose connection we have now to analyse.

Assume any point O′ opposed to the angle at O, and draw from O′ perpendiculars O′A′, O′B′ and O′C′, to the three planes which contain the given solid angle.

Fig. 201.



The solid angles at O and O′, which are respectively contained by the three lines diverging from those points, are "opposite angles," and fulfil the equations 40 and 41.

To demonstrate this assertion we need only remark that from art. 63—3, the plane which passes through O′A′, O′B′ will be at right angles to each of the planes to which these lines are perpendiculars, and consequently to their common intersection; for as a similar remark applies to the plane that passes through the perpendiculars O′B′ and O′C′, it will follow that OCB′ and OAB′ are right angles, and consequently, that AB′C is the supplement of AOC, or that

$$B' = 180 - b.*$$

The perfect identity of the construction with respect to the planes of a, b and c will render whatever is proved for one of these angles, applicable to the others; and hence we derive the equations

* See the notation of the preceding page.

$$A' = 180 - a$$
$$C' = 180 - c$$

which are equivalent to 41.

But since O′B′ and O′C′ are perpendiculars to the planes whereon they fall, the angles at B′ and C′ are right angles, and B′O′C′ is the supplement of B′AC′. This result, expressed by the notation we have adopted, gives

$$a' = 180 - A$$

whence, from the identity observed in the construction, we derive

$$b' = 180 - B$$
$$c' = 180 - C.$$

The demonstration here given of the equations 40 and 41 may be derived more immediately from a subsequent article; but the reasoning will not be understood until the intermediate analysis has been read, and should therefore be omitted until the student has made the necessary progress.

The demonstration alluded to is as follows:

By the subsequent article above mentioned, perpendiculars to planes include angles equal to those at which the planes are inclined.

If, therefore, a, b and c are plane angles, the lines p, p' and p'' respectively perpendicular to the planes of a, b and c, will include angles equal to the inclinations of those planes.

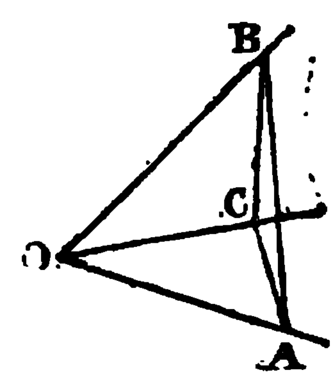
But, reciprocally, the lines that form the angles a, b and c are perpendicular to the planes passing through p and p', p and p'', and p' p''; and must form angles equal to the inclinations of the latter.

"If, then, three lines are the edges of a solid angle,

the perpendiculars to the planes in which they lie will be the edges of a second solid angle, so related to the first, that a plane angle in the one solid will be an inclination of two planes in the other."

But* the inclinations of planes that form a solid angle are measured in the same order and directions as the inclinations of the sides of a closed figure: the inclinations of the planes that form the angle at O, for example, being measured in the order and directions of the inclinations of the sides which form the closed figure CBA: hence, by the subsequent article referred to, the angles of which we have spoken, are not the interior angles of the solid, but their supplements: and when the interior angles are used, the relations of the two solid angles above mentioned, will be changed, and the proposition will be expressed as follows:

Fig. 202.



"If three lines are the edges of a solid angle, the perpendiculars to the planes in which they lie will be the edges of a second solid angle, so related to the first, that a plane angle in the one solid will be the supplement of an inclination of two planes in the other."

Which agrees with the proposition of page 332.

From inspecting the equations 40 and 41, as well as from the demonstrations of them since given, it will be seen that of two solid angles A and B, if B is the "opposite angle" to A, reciprocally A is the opposite angle to B.

Every solid angle is therefore an "opposite angle" to some other; and, hence, whatever be proved true of all "opposite solid angles," must be true of all solid angles whatever.

* See note 7.

Sect. III. Relations of three divergent lines.
Art. 146. Use of opposite solid angles in analysis.

146. These properties will readily enable us to transform the equations 39; for writing

$$a = \tfrac{1}{2} - A', \ b = \tfrac{1}{2} - B', \ c = \tfrac{1}{2} - C'$$
$$A = \tfrac{1}{2} - a', \ B = \tfrac{1}{2} - b', \ C = \tfrac{1}{2} - c',$$

substituting these values in the equations 39, and, finally, omitting the accents, those equations become

$$\begin{aligned} \cos. A &= \sin. B \sin. C \cos. a - \cos. B \cos. C \\ \cos. B &= \sin. A \sin. C \cos. b - \cos. A \cos. C \quad \ldots 42 \\ \cos. C &= \sin. A \sin. B \cos. c - \cos. A \cos. B. \end{aligned}$$

147. The equations of art. 142, if written according to the notation we have since employed, will be

$$\begin{aligned} &32^* \ \ldots\ldots \ \cos. c = \cos. b \cos. a \\ &33^* \ \ldots\ldots \ \sin. b = \sin. c \sin. B \\ &34^* \ \ldots\ldots \ \cos. B = \sin. A \cos. b \\ &35^* \ \ldots\ldots \ \sin. a = \tan. b \cot. B \\ &36^* \ \ldots\ldots \ \cos. c = \cot. B \cot. A \\ &37^* \ \ldots\ldots \ \cos. B = \cot. c \tan. a. \end{aligned}$$

These equations were demonstrated in the article alluded to, with the exception of the third and last; which follow, however, from the sets of equations 39 and 42.

For the value of cos. B given in the last set, reduces, when $\tfrac{1}{4}$ is substituted for C, to the value given in the expression 34*, and if the same substitution is made in the last equation of 39, it becomes

$$\cos. c = \cos. a \cos. b,$$

and since from 33* we deduce, by analogy,

$$\sin. c = \frac{\sin. a}{\sin. A};$$

there will arise, dividing the first of these results by the second,

$$\text{cot. } c = \text{cot. } a \text{ cos. } b \text{ sin. A}:$$

Now, from the equation 34*,

$$\text{cos. } b \text{ sin. A} = \text{cos. B}:$$

hence,

$$\text{cot. } c = \text{cot. } a \text{ cos. B},$$

or,

$$\text{cos. B} = \text{cot. } c \text{ tan. } a;$$

which is the theorem to be demonstrated.

148. These formulæ, which we have deduced for the case wherein one of the *solid* angles is a right angle, will also apply where a *plane* angle is right; or, in other words, when two of the divergent lines are inclined to each other at an angle of 90°; as will appear evident by employing the sub-contrary angle. For since the solid angles of the latter are supplementary to the plane angles of the solid whose properties we are investigating, it will follow, that a solid right angle in one corresponds to a plane angle in the other.

Assuming, for example,

$$\text{A, B, C, and } a, b, 90$$

to be the angles of the primitive solid, we shall have

$$\text{A}' = 180 - a,\ \text{B}' = 180 - b,\ \text{C}' = 90$$
$$a' = 180 - \text{A},\ b' = 180 - \text{B},\ c' = 180 - \text{C},$$

for those of its sub-contrary.

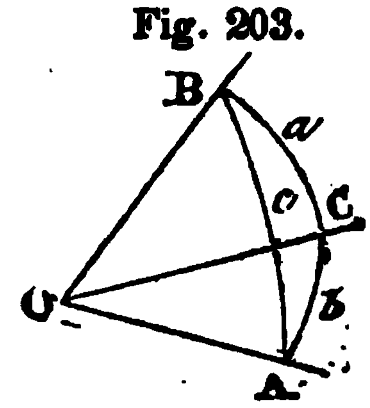
And as the expressions here given for changing the parts of the primitive into those of the sub-contrary do not alter the *values* of the trigonometrical ratios of the

formulæ wherein they are substituted, the equations of the preceding article will equally apply to a subcontrary with a "solid," or a primitive with a "plane" right angle; in other words, they will apply whether the right angle is plane or solid. It must be borne in mind, however, that although the substitution in question does not alter the *values* of the ratios, it frequently alters their signs, and the result must be written accordingly.

149. The extent and frequent use of the formulæ, render it important to retain them in the memory, and as this cannot, under their present form, be accomplished without undue exertion of that faculty, it is customary to employ an arrangement, founded not on the nature of the subject, but on the laws of suggestion.

The following rules, altered from those given by the celebrated inventor of logarithms, are very convenient for this purpose.

Fig. 203.



Arrange the six parts,

A, c, B, a, C, b

in the order which they occupy when regarded as relations of the closed figure ABC.

Omit the part which is ninety, and any one of the parts remaining will, then, have two that are *adjacent*, and two that are opposite to it.

This done, and provided that we substitute the complements of the parts opposite to the 90, instead of the parts themselves, the formulæ of art. 147 will all be included in the following

RULE.

Assuming either of the five relations as a middle part,

the sine of the middle part will be equal to the product of the *cosines* of the *opposite*, or the *tangents* of the adjacent parts.

150. The equations 39 containing all the relations of three divergent lines, reduce the investigation to an algebraic process of elimination; and the subsequent part of the inquiry will therefore resemble the course pursued with regard to three lines which form a closed figure; but as the form of the equations is not the same as that of the equations 1, we must have recourse to other processes to disengage the parts sought, and to express them in terms of those parts which are given.

In the case before, as in that alluded to, the number of relations is six, and we must therefore have three things given to determine the remainder.

The data, however, are not subject to the restriction which limited the given parts in the former problem, any three of the six things being data sufficient to render the problem determinate.

This fact may be sufficiently seen from the form of the equations 39 and 42, which is such that, except for particular values of the parts, neither the numerator nor denominator of the fraction given by Bezout's rule, will be equal to zero.

The analogy between the case we are considering and that of art. 51, will be seen in other particulars; and, as in that case, we must conduct our eliminations with an especial view to the form of the result, which, unless it is either monomial or consists of factors of the first degree, will not permit the use of logarithms. The expressions we have deduced for the partial cases already considered,

are of this form; and it now remains for us to deduce, in the most general case of three divergent lines, values for each of the parts, that shall fulfil the two conditions to which we have alluded, namely, that shall be expressed in terms of three other parts, and that shall either be monomial, or divisible into factors of the first degree.

A very simple analysis, by auxiliary elements, will enable us to avoid a tedious elimination from the equations in question, and lead to a result having the required form.

Fig. 204.

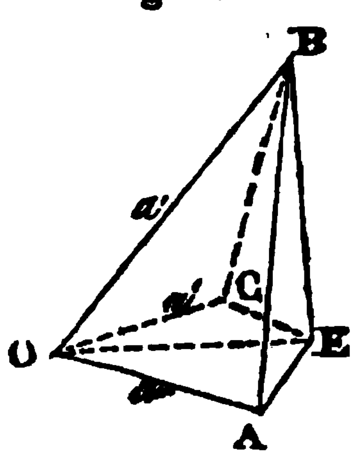
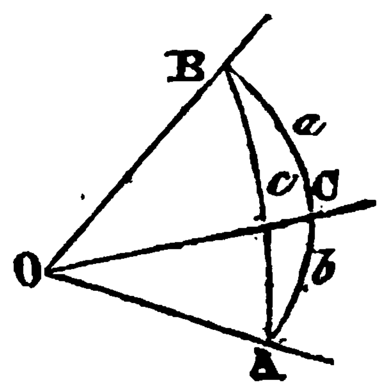


Fig. 205.



This analysis consists in equating the values of BE, a common side of the two rectangular pyramids employed in establishing the equations 39.

Denoting this common side by p, we should have, by applying equation 33* to one of the pyramids,

$$\sin. p = \sin. c \sin. A;$$

and by applying it to the other,

$$\sin. p = \sin. a \sin. C:$$

whence, equating their values, and dividing both sides by $\sin. c \sin. C$, there arises,

$$\frac{\sin. a}{\sin. c} = \frac{\sin. A}{\sin. C}.$$

The results deduced for the sides a and c will evidently apply to any other sides, and thus we conclude generally, that,

The ratio of the sines of two sides is equal to the ratio of the sines of two angles of the same name.

A result expressed by the equations

Chap. I. Detailed analysis of the relations of direction; and of the relations peculiar to three, to four, and to a greater number of points.
Art. 150. In the most general case three parts determine the remainder.
Art. 151. Enumeration of data.

$$\left.\begin{aligned} \frac{\sin.\ a}{\sin.\ b} &= \frac{\sin.\ A}{\sin.\ B} \\ \frac{\sin.\ a}{\sin.\ c} &= \frac{\sin.\ A}{\sin.\ C} \\ \frac{\sin.\ b}{\sin.\ c} &= \frac{\sin.\ B}{\sin.\ C}. \end{aligned}\right\} \ldots\ldots 43$$

151. The combinations, three and three, that can be made of the six relatives of three divergent lines, will express the different forms under which the data may appear; but as some of the cases may be solved by analogy from others, it is only necessary to consider certain of the combinations as distinct: they are conveniently arranged as follows:

DATA.

1. Two plane angles, and a solid angle opposite to one of them.
2. Two solid angles, and a plane angle opposite to one of them.
3. Two plane angles, and a solid angle included between them.
4. Two solid angles, and a plane angle adjacent to both.
5. Three plane angles.
6. Three solid angles.

152. As the equations 43 suffice for the solution of the two first of these cases, we may proceed to consider the third.

The equations 39, were they under a form that

permitted the ready application of logarithms, would render unnecessary any further investigation of the case in question; but as these expressions are not adapted to that species of calculation, it will be necessary to replace them by others that are the products of factors of the first degree.

The reduction is too long to permit a detailed explanation of the *views* which guide the analysist in the successive steps, but the process itself will be readily understood.

Substituting for cos. c, in the first of the equations 39, its value obtained from the last, there arises,

$$\cos. a = \cos. A \sin. b \sin. c + \sin. a \sin. b \cos. b \cos. C + \cos. a \cos. b^2;$$

or, transposing, substituting for $1 - \cos. b^2$ its value $\sin. b^2$, and dividing by $\sin. b$,

$$\cos. A \sin. c = \cos. a \sin. b - \sin. a \cos. b \cos. C.$$

This equation gives, by analogy,

$$\cos. B \sin. c = \cos. b \sin. a - \sin. b \cos. a \cos. C,$$

And adding the two together, and substituting for $\sin. a \cos b + \cos. a \sin. b$ its value $\sin. (a + b)$

$$\{\cos. A + \cos. B\} \sin. c = \{1 - \cos. C\} \sin. (a + b) \quad . . \alpha$$

But we have, art. 150,

$$\frac{\sin. A}{\sin. C} = \frac{\sin. a}{\sin. c}$$

$$\frac{\sin. B}{\sin. C} = \frac{\sin. b}{\sin. c},$$

and, adding and subtracting these equations, there results,

$$\{\sin. A + \sin. B\} \sin. c = \sin. C \{\sin. a + \sin. b\}$$
$$\{\sin. A - \sin. B\} \sin. c = \sin. C \{\sin. a - \sin. b\}$$

which, divided by the equation α, produce

$$\frac{\sin. A + \sin. B}{\cos. A + \cos. B} = \frac{\sin. C}{1 - \cos. C} \cdot \frac{\sin. a + \sin. b}{\sin.(a + b)} \quad \ldots \beta$$

$$\frac{\sin. A - \sin. B}{\cos. A + \cos. B} = \frac{\sin. C}{1 - \cos. c} \cdot \frac{\sin. a - \sin. b}{\sin.(a + b)} \quad \ldots \gamma$$

But referring to the table of trigonometrical formulæ, art. 126, we observe that,

$$\frac{\sin. a \pm \sin. b}{\cos. a + \cos. b} = \tan. \tfrac{1}{2} (a \pm b)$$

$$\frac{\sin. a + \sin. b}{\sin.(a + b)} = \frac{\cos. \frac{1}{2} (a - b)}{\cos. \frac{1}{2} (a + b)}$$

$$\frac{\sin. a - \sin. b}{\sin. (a + b)} = \frac{\sin. \frac{1}{2} (a - b)}{\sin. \frac{1}{2} (a + b)}$$

$$\frac{\sin. a}{1 - \cos. a} = \cot. \tfrac{1}{2} a$$

Where the angles a and b are taken at pleasure, and have no relations to the angles of the same name in the particular problem we are considering: making them successively to coincide with these, and with the angles A, B and C, the equations β and γ reduce to

$$\tan. \tfrac{1}{2} (A + B) = \frac{\cos. \frac{1}{2} (a - b)}{\cos. \frac{1}{2} (a + b)} \cot. \tfrac{1}{2} C \quad . \ . \ 44$$

$$\tan. \tfrac{1}{2} (A - B) = \frac{\sin. \frac{1}{2} (a - b)}{\sin. \frac{1}{2} (a + b)} \cot. \tfrac{1}{2} C \quad . \ . \ 45$$

expressions that are well adapted to logarithms, and which enable us, when two plane angles and the included solid angle are known, to determine both the sum and the difference of the remaining solid angles, and thence the angles themselves.

153. In the solid angle which is opposite to that we

have been considering, there will be given two solid angles, and a plane angle interjacent to them; but substituting in the preceding formulæ, $\frac{1}{2} - a'$ for A, $\frac{1}{2} - b'$ for B, &c. and neglecting the accents, these expressions become,

$$-\tan.\ \tfrac{1}{2}\ (a + b) = \frac{\cos.\ \frac{1}{2}\ (A - B)}{\cos.\ \frac{1}{2}\ (A + B)}\ \tan.\ \tfrac{1}{2}\ c \quad . \ . \ . \ 46$$

$$-\tan.\ \tfrac{1}{2}\ (a - b) = \frac{\sin.\ \frac{1}{2}\ (A - B)}{\sin.\ \frac{1}{2}\ (A + B)}\ \tan.\ \tfrac{1}{2}\ c \quad . \ . \ . \ 47$$

which enable us to determine the remaining plane angles.

154. Although the formulæ of the two preceding articles are perhaps the most convenient that can be used in solving the thirdand fourth cases of art. 151, we ought not, when estimating their practical utility, to lose sight of the length of the investigation: rules that are to be employed in particular cases should be readily deduced from those general principles of analysis that apply to all cases, and which owe their chief value to the relief they afford the memory.

Estimated in this point of view, our analysis, by auxiliary elements, will appear superior to those which have occupied us in the articles alluded to.

Employing the same figure, and assuming the given parts to be the solid angle A, and the plane angles b and c, we have in the rectangular pyramid which contains the angle A, two parts given; whence, by the known properties of that solid, all the remaining relations of it can be found.

Fig. 206.

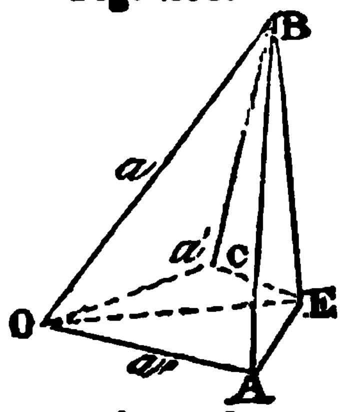
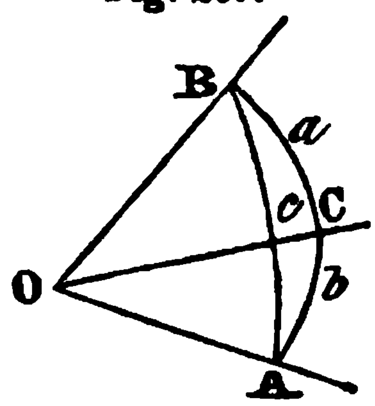


Fig. 207.



The angles at the bases of these two pyramids, as they are merely segments of the angle b, are not contained in the system of notation which we have adopted, art. 144, for the whole triangle; we will express them by s and s', and represent, as before, the perpendicular CE by p.

The equation 37* will give

$$\cos. A = \cot. c \tan. s;$$

whence,

$$\tan. s = \cos. A \tan. c: \quad . \; . \; . \; . \; 48$$

From this expression the value of s is readily determined, and thence we deduce the value of s' by the equation

$$s' = b - s. \quad . \; . \; . \; . \; 49$$

Again, from the same pyramid, we have, 32*,

$$\cos. c = \cos. p \cos. s.$$

And in the remaining pyramid,

$$\cos. a = \cos. p \cos. s'$$

And dividing the last equation by the last but one,

$$\frac{\cos. a}{\cos. c} = \frac{\cos. s'}{\cos. s} \quad . \; . \; . \; . \; 50.$$

an expression that enables us to determine the value of the unknown angle a, and thence by the cases 1 and 2, art. 151, the remaining parts sought.

The analysis here employed follows so immediately from the general principles of the science, that should the result be forgotten it is easy for any one who has fully comprehended the principles of geometrical investigation, to rediscover it for himself; the equation 48 is, indeed, merely a particular application of the formulæ for the rectangular pyramid; and the equation 50 is too simple to escape the memory; it may be expressed as the following rule:

If a solid angle formed by three divergent lines is divided into two rectangular pyramids, the ratio of the cosines of the undivided angles is equal to the ratio of the cosines of the adjacent bases.

When this analysis of the third case is employed, the method of deducing the solution of the fourth case, proceeds, as in art. 153, upon the use of the subcontrary solid angle: as the equation 50, however, contains parts that are foreign to the relations of three divergent lines, we have no assurance, from what has hitherto been demonstrated, that the relations of the subcontrary angle can be applied to them; and hence, in place of substituting in these equations, as in the equations 44 and 45, the supplements of the several parts, it is necessary to proceed upon a more certain foundation; and by means of the equations

$$a=\tfrac{1}{2}-A',\ b=\tfrac{1}{2}-B',\ c=\tfrac{1}{2}-C',\ A=\tfrac{1}{2}-a',\ B=\tfrac{1}{2}-b',$$
$$C=\tfrac{1}{2}-c' \quad . \ . \ . \ . \ . \ (a)$$

transform the solid investigated into its subcontrary solid.

The parts given in the latter will then fall under the case (3) already solved; and determining by that case the remaining parts of the subcontrary solid angle, we can, from these, by means of the equations a, deduce the relations sought.

Or dividing one of the known solid angles, C, by a perpendicular drawn to the opposite plane angle, c, and calling the segments of the former (the solid angle) S and S′, we can derive, by an analysis in every respect similar to the preceding, the equations,

2 T

$$\cot. S = \cos. a \tan. B$$
$$S' = C - S$$
$$\frac{\cos. A}{\cos. B} = \frac{\sin. S'}{\sin. S}$$

155. The first of the equations, 39, gives,

$$\cos. A = \frac{\cos. a - \cos. b \cos. c}{\sin. b \sin. c} \quad \ldots\ldots \beta$$

an expression that must be reduced to factors in order to solve the fifth case, art. 151.

The denominator is already composed of simple factors, and it only remains, therefore, to bring the numerator to the same form.

Examining the table, art. 126, we observe the sums and differences of the cosines to be among the simplest quantities of that kind which can be expressed in factors.

Now the expression,

$$\cos. a - \cos. b \cos. c$$

consists of a compound quantity subtracted from a cosine; and if, therefore, we can in any way substitute for the second, or compound term, the cosine of a single angle, the expression we are considering will take the form desired.

But the second term,

$$\cos. b \cos. c$$

taken in conjunction with the denominator of the fraction β, will be expressed as a cosine of a single angle, since

$$\cos. b \cos. c + \sin. b \sin. c = \cos. (b - c).$$

The first step in the inquiry, then, is to bring the denominator of the fraction α into the numerator, that we may combine it with the term cos. b cos. c.

This may be accomplished either by adding or subtracting unity to both sides of the equation: let us choose the latter; there will arise, by changing the signs,

$$1 - \cos. A = 1 - \frac{\cos. a - \cos. b \cos. c}{\sin. b \sin. c}$$

or,

$$1 - \cos. A = \frac{\cos. (b - c) - \cos. a}{\sin. b \sin. c}$$

But taking any two angles, m and n, the formulæ of the table give

$$\cos. m - \cos. n = 2 \sin. \tfrac{1}{2} (m + n) \sin. \tfrac{1}{2} (m - n)$$

which, making $m = b - c$, and $n = a$, transforms our last equation into

$$1 - \cos. A = \frac{2 \sin. \frac{1}{2} (a + b - c) \sin. \frac{1}{2} (a + c - b)}{\sin. b \sin. c}.$$

The right hand member is now under the form required, but that on the left hand is not adapted to the use of logarithms. Referring, however, to the table of trigonometric formulæ of which such frequent use has been made in the preceding pages, we observe, that

$$1 - \cos. A = 2 \sin. \tfrac{1}{2} A^2$$

and adopting this substitution, the equation we are considering becomes

$$\sin. \tfrac{1}{2} A^2 = \frac{\sin. \frac{1}{2} (a + b - c) \sin. \frac{1}{2} (a + c - b)}{\sin. b \sin. c}$$

an expression that would serve all the purposes of calculation.

It admits, however, of some further reduction; for since

$$\tfrac{1}{2} \{a + c - b\} = \tfrac{1}{2} \{a + b + c\} - b$$

and

$$\tfrac{1}{2} \{a + b - c\} = \tfrac{1}{2} \{a + b + c\} - c$$

by putting $a + b + c = s$, we shall finally obtain

$$\sin. \tfrac{1}{2} A = \sqrt{\left\{\frac{\sin. (\frac{1}{2} s - b) \sin. \{\frac{1}{2} s - c\}}{\sin. b \sin. c}\right\}} \quad . \; . \; . \; 51$$

an expression very analogous to the equation (27) obtained for the similar case in plane trigonometry.

When, by means of this formula, one of the solid angles has been found, the remainder of the computation will be completed from the equations 43; and we may, therefore, now proceed to consider the last case enumerated in art. 151.

156. But the use of the subcontrary solid angle renders a detailed investigation of this case unnecessary; since, substituting for a, b and c their values,

$$A = \tfrac{1}{2} - a', \; b = \tfrac{1}{2} - B', \; c = \tfrac{1}{2} - C';$$

reducing, and afterwards omitting the accents, the expression 51 becomes,

$$\cos. \tfrac{1}{2} a = \sqrt{\left\{\frac{\cos. (\frac{1}{2} S - B) \cos. (\frac{1}{2} S - C)}{\sin. B \sin. C.}\right\}} \quad . \; . \; . \; 52$$

where S expresses the sum of the three solid angles, A, B and C.

The value of a obtained from the equation 52, the remainder of the computation is completed as in the preceding article.

157. The formulæ 51 and 52, although affected with the double sign common to all irrational quantities, will present no ambiguity; for as the value of half an angle cannot be greater than $\frac{1}{4}$, we are always, when using these formulæ, to make choice of the angle found in the tables.

The same remark, however, does not apply to the solutions obtained from the formulæ 43, 44, or 46, which, as in the similar cases of plane trigonometry, admit a choice between the angle of the table and its supplement: indeed, for a reason that will hereafter be noticed, the analogy is so perfect, that we may apply, in both cases, the same method of investigation, and, therefore, remove the ambiguity to which allusion is made, by having recourse to those properties of three divergent lines, that are found in every problem concerning the latter.

PROPERTIES COMMON TO EVERY CASE OF THREE DIVERGENT LINES.

1. The sum of any two of the plane angles is greater than the third.

The demonstration of this property has been given in art. 140, and there follows from it, as in art. 135, that any side

$$a < \frac{a+b+c}{2}$$

and,

$$> \frac{a+b+c}{4}$$

2. The sum of the three plane angles is less than unity.

For, by producing the lines that contain the angles a and b, we form a second solid angle, which is contained by the angles $\frac{1}{2} - a$, $\frac{1}{2} - b$, and c; and as any two of these are greater than the third, we have,

$$(\tfrac{1}{2} - a) + (\tfrac{1}{2} - b) > c,$$

or,

$$(a+b)+(\tfrac{1}{2}-a)+(\tfrac{1}{2}-b)>a+b+c,$$

that is,

$$1>a+b+c.$$

3. The sum of the three solid angles is greater than $\frac{1}{2}$, and less than $\frac{3}{2}$.

Each solid angle is less than two right angles, and therefore the sum of the three angles is less than six right angles, or than $\frac{3}{2}$.

And since the measure of each solid angle is equal to $\frac{1}{2}$, minus the corresponding plane angle of the subcontrary solid, it will follow that in every case of three divergent lines, the sum of the three solid angles has for its measure $\frac{3}{2}$, minus the sum of the three plane angles of the subcontrary solid. But this last sum is less than unity, and, consequently, by subtracting it from $\frac{3}{2}$, the remainder will be greater than $\frac{1}{2}$.

4. The sum of any two solid angles is greater than the supplement of the third angle.

For since it appears in the demonstration of the second of the properties here enumerated, that

$$(\tfrac{1}{2}-a)+(\tfrac{1}{2}-b)>c$$

we shall have in the subcontrary solid,

$$A'+B'>\tfrac{1}{2}-C'$$

The greater solid angle is opposite the greater plane angle; and the converse.

5. These results may be established, as in art. 135, by means of the equation

$$\frac{\sin.\ A}{\sin.\ B}=\frac{\sin.\ a}{\sin.\ b},$$

but the reasoning is somewhat more complex, and may

be greatly simplified by having recourse to the equation, art. 153,

$$\tan. \tfrac{1}{2} (\mathrm{A} - \mathrm{B}) = \frac{\sin. \tfrac{1}{2} (a \quad b)}{\sin. \tfrac{1}{2} (a + b)} \cot. \tfrac{1}{2} \mathrm{C}.$$

For attending to the limits of the semi-angles which enter this formula, we observe, that since $\tfrac{1}{2} (a + b)$ is less than $\tfrac{1}{2}$, its sine must be positive; and since the remaining semi-angles are less than $\tfrac{1}{4}$ their sines, tangents and co-tangents can only become negative by the angle itself having a negative value: moreover, the sides of an equation have the same algebraic sign, and hence, if a is greater or less than b, the angle A must also be greater or less than B, and the converse.

6. If the sum of any two solid angles is equal to, greater, or less than $\tfrac{1}{2}$, the sum of the opposite angles must also be equal to, greater, or less than $\tfrac{1}{2}$; and conversely. Or, as it is sometimes expressed, The sum of any two sides must be of the same kind, with respect to 180, as the sum of the opposite angles.

This result follows immediately from the equation 44, art. 153,

$$\tan. \tfrac{1}{2} (\mathrm{A} + \mathrm{B}) = \frac{\cos. \tfrac{1}{2} (a - b)}{\cos. \tfrac{1}{2} (a + b)} \cot. \tfrac{1}{2} \mathrm{C}$$

For as $\cot. \tfrac{1}{2}$ C and $\cos. \tfrac{1}{2} (a - b)$ are necessarily positive, the signs of $\tan. \tfrac{1}{2} (\mathrm{A} + \mathrm{B})$ and $\cos. \tfrac{1}{2} (a + b)$ must change together; or, in other words, if A + B is greater than $\tfrac{1}{2}$, the plane angle $a + b$ must also be greater than $\tfrac{1}{2}$, and when A + B is less than $\tfrac{1}{2}$, the sum of the two plane angles must be so likewise; the remainder of the proposition is a necessary result from that which has been demonstrated.

PARTICULAR RELATIONS OF THREE DIVERGENT LINES.

158—1. Three divergent lines, whose mutual inclinations are equal, have also the mutual inclinations of their planes equal; and the converse.

For if a, b and c are equal, it follows from the equations

$$\frac{\sin. A}{\sin. B} = \frac{\sin. a}{\sin. b}, \quad \frac{\sin. A}{\sin. C} = \frac{\sin. a}{\sin. c},$$

that A, B and C are also equal; and the converse.

2. If two of the plane angles are equal, the opposite solid angles are also equal.

This follows from the same equations.

159. The use which, in art. 157, it is proposed to make of the properties common to all cases of three divergent lines, can often be replaced by another process. A judicious use of the signs attached to the trigonometrical ratios, will remove the ambiguity in every instance except that where the answer is obtained through the medium of its sine. Assuming, for example, that in a problem solved by the formula, art. 154,

$$\frac{\cos. a}{\cos. b} = \frac{\cos. s'}{\cos. s}.$$

the angles s' and s were less, and the angle b greater than 90; the cosines of the former would, in this case, be positive, and the cosine of the latter negative; and attaching to each ratio its appropriate sign

$$\frac{\cos. a}{\overset{-}{\cos. b}} = \frac{\overset{+}{\cos. s'}}{\overset{+}{\cos. s}}$$

the known laws of equations will require that cos. a should be negative, and the angle a, consequently, greater than 90.

In employing this rule, however, we must not forget the various significations that may be attached to a negative ratio.

A negative tangent may denote either a negative angle, or an angle that exceeds 90, and we must judge from the nature of the problem which of these interpretations ought to be adopted.

In the problem, for example, which is discussed in art. 154, if the angle A is obtuse, the tangent of s will be negative; but this result is to be understood, not as implying a value of s greater than 90, but a negative value: for as the point E will, on the hypothesis adopted, fall without A, the direction of s will be changed, and the formula

$$s' = b - s$$

become

$$s' = b + s.$$

A remarkable case of this problem occurs when both A and b, art. 154, are obtuse; for, notwithstanding the tangent of s is then positive, the angle to be taken for s may be either that found in the tables, or its supplement: in fact, the tangent of s will then be negative on two accounts, namely, as the tangent of a negative angle, and of an angle which exceeds 90°; the resulting value will therefore be positive, and as the theory of correlation, sect. III., teaches us that changes in the signs will ac-

company those in the figure, we conclude that it is permitted to assume s as lying in a *positive* direction, and *less* than 90°, or as lying in a *negative* direction, and *greater* than 90.

After what has been said respecting the choice between the angle of the tables and its supplement, we need only add that cases 1 and 2 of art. 151, will often allow of two solutions.

A result that may be deduced from the formula

$$\frac{\sin. A}{\sin. B}=\frac{\sin. a}{\sin. b}:$$

for whether we use in this expression A, or, 180—A, the equality will still obtain; and as A or its supplement may both be greater, or both less, than B, the condition relative to the greater side and angle may be fulfilled by either of these values.

A similar remark applies when 180—a is substituted for a; and, thus, in either of the two first cases enumerated in the article alluded to, we may fall upon a problem that admits of two solutions; but such a result, it appears, will not happen unless both the angle sought and its supplement are of the *same kind*, art. 157, with respect to one of the given angles;—namely, to the given solid angle when the angle sought is solid, and to the given plane angle when the angle sought is plane.

APPENDIX TO SECTION III.

EXAMPLES.

1. The solid angle S is bounded by the three plane angles

$$a = 15^\circ,\ b = 77^\circ\ 12',\ c = 90;$$

required the solid angles A, B and C.

To find C.

Writing the parts in order, and omitting the 90, we have

$$A,\ B,\ a,\ C,\ b,\ A,$$

where C is adjacent to a and b.

Hence by Napier's rules,

$$-\ \overset{-}{\cos.}\ C = \overset{+}{\cot.}\ a\ .\ \overset{+}{\cot.}\ b$$

And

$$\begin{array}{rl}
\log.\ \overset{-}{\cos.}\ C. &= \log.\ \overset{+}{\cot.}\ a + \log.\ \overset{+}{\cot.}\ b - 10 \\
\log.\ \cot.\ 15^\circ &= 10.5719475 \\
\log.\ \cot.\ 77^\circ\ 12' &= \underline{\ \ 9.3563977} \\
&= 19.9283452 \\
&\ \ \ \ \underline{10\qquad\qquad} \\
\log.\ \cos.\ \underline{32^\circ\ \ 1'} &= \ \ 9.9283452 \\
C = 147\ \ 59' &
\end{array}$$

Appendix to Section III.

Examples.

The negative sign which is made in this example to precede the term cos. C is a part, art. 148, of the formula, and arises from the use of the subcontrary angle.

To find A.

Referring to the preceding arrangement of the parts, we see that a is opposite both to b and A, and, therefore,

$$-\cos. a = \sin. b \times (-\cos. A)$$

or

$$\overset{+}{\cos}. A = \frac{\overset{+}{\cos}. a}{\overset{+}{\sin}. b}$$

$$\log. \cos. A = \log. \cos. a + \text{ar. co.} \log. \sin. b + 10$$

$$\begin{aligned} \log. \cos. 15° &= 9.9849438 \\ \text{ar. co.} \log. \sin. 77° 12' &= \overline{10}.0109289 \\ &\overline{\overline{1}.9958727} \\ &10 \\ \log. \cos. 7° 53' &= \overline{9.9958727} \\ A &= 7° 53. \end{aligned}$$

To find B.

$$-\cos. b = -\cos. B . \sin. a.$$

or

$$\overset{+}{\cos}. B = \frac{\overset{+}{\cos}. b}{\overset{+}{\sin}. a}$$

$$\log. \cos. B = \log. \cos. b + \text{ar. co.} \log. \sin. a + 10$$

$$\begin{aligned} \log. \cos. 77° 12' &= 9.3454688 \\ \text{ar. co.} \log. \sin. 15 &= \overline{10}.5870038 \\ &= \overline{\overline{1}.9324726} \\ &10 \\ \log. \cos. 31° 8' &= \overline{9.9324726} \\ B &= 31° 8' \end{aligned}$$

Appendix to Section III.
Examples.

2. Three divergent lines have the following relations of their parts:

$$c = 65° \, 5', \ A = 48° \, 12', \ C = 90;$$

required the remaining relations.

The parts, arranged in order, are

$$A, \ b, \ a, \ B, \ c, \ A$$

To find a.

$$\sin. a = \sin. A \sin. c$$

$$\log. \sin. a = \log. \sin. A + \log. \sin. c - 10$$

$$\begin{array}{lcl} \log. \sin. 48° \, 12' & = & 9.8724337 \\ \log. \sin. 65° \, \underline{5'} & = & \underline{9.9575697} \\ \overline{10} & = & \overline{10} \\ \log. \sin. 42° \, 13' \, 19'' & = & \overline{9.8300034} \\ a = 42° \, 13' \, 19'' & & \end{array}$$

The side a is known to be acute by the property that the greater side is opposite the greater angle.

To find b.

$$\overset{+}{\cos.} A = \overset{+}{\tan.} b \overset{+}{\cot.} c$$

or

$$\tan. b = \cos. A \tan. c$$

$$\begin{array}{lcl} \log. \cos. 48° \, 12 & = & 9.8238213 \\ \log. \tan. 65 \quad \underline{5} & = & \underline{10.3329786} \\ \overline{10} & = & \overline{10} \\ \log. \tan. 55° \quad 7' \, 32'' & = & \overline{10.1567999} \\ b = 55° \quad 7' \, 32'' & & \end{array}$$

To find B.

$$\overset{+}{\text{cos.}}\ c = \overset{+}{\text{cot.}}\ A\ \overset{+}{\text{cot.}}\ B$$

or

$$\text{cot. } B = \text{cos. } c. \text{ tan } A$$

$$\begin{array}{lcl}
\text{log. cos. } 65^\circ\ \ 5' & = & 9.6245911 \\
\text{log. tan. } 48^\circ\ 12' & = & 10.0486124 \\
\qquad\qquad\quad \overline{10} & = & \overline{10} \\
\text{log. cot. } 64^\circ\ 46'\ 14'' & = & \overline{\ 9.6732035} \\
\quad B = 64^\circ\ 46'\ 14'' & &
\end{array}$$

3. Three divergent lines have the following relations of their parts,

$$a = 67^\circ\ 35',\ b = 104^\circ\ 16'\ 20'',\ C = 81^\circ\ 8'\ 11'';$$

required the remaining relations.

Letting fall a perpendicular p upon the angle b; calling the segments s and s'; and arranging in order the parts of one of the rectangular pyramids so formed

$$a,\ C,\ s,\ p,\ \text{seg. of B},\ a;$$

we shall have

$$\overset{+}{\text{cos.}}\ C = \overset{+}{\text{tan.}}\ s\ \overset{+}{\text{cot.}}\ a.$$

or

$$\text{tan. } s = \text{cos. } c \text{ tan. } a$$

$$\begin{array}{lcl}
\text{log. cos. } 81^\circ\ \ 8'\ 11'' & = & 9.1877543 \\
\text{log. tan. } 67^\circ\ 35'\ \ 0 & = & 10.3845649 \\
\qquad\qquad\qquad\quad \overline{10} & = & \overline{10} \\
\text{log. tan. } 20^\circ\ 28'\ 55'' & = & \overline{\ 9.5723192} \\
\quad b = 104^\circ\ 16'\ 20'' & & \\
\quad s = \ \ \underline{20\ \ 28\ \ 55} & & \\
b - s = s' = \ \ 83\ \ 47\ \ 25 & &
\end{array}$$

Appendix to Section III.
Examples.

But

$$\frac{\overset{+}{\cos.\, c}}{\overset{+}{\cos.\, a}} = \frac{\overset{+}{\cos.\, s'}}{\overset{+}{\cos.\, s}}.$$

Log. cos. c = log. cos. a + log. cos. s' + ar. co. log. cos. s

$$\begin{array}{rl}
\text{log. cos. } 67^\circ\ 35'\ 00 = & 9.5813116 \\
\text{log. cos. } 83\ \ 47\ \ 25 = & 9.0340986 \\
\text{ar. co. log. cos. } 20\ \ 28\ \ 55 = & \overline{10}.0283613 \\
\hline
\text{log. cos. } 87\ \ 28\ \ 35 = & 8.6437715 \\
c = 87^\circ\, 28'\, 35''. &
\end{array}$$

To find A.

The parts of the second rectangular pyramid are,

A, c, seg. of B, p, s', A;

hence,

$$\overset{+}{\cos.}\ \text{A} = \overset{+}{\tan.}\ s', \overset{+}{\cot.}\ c$$

$$\begin{array}{rl}
\text{log. tan. } 83^\circ\ 47'\ 25'' = & 10.9633457 \\
\text{log. cot. } 87\ \ 28\ \ 35 = & 8.6441786 \\
\overline{10} = & \overline{10} \\
\hline
\text{log. cos. } 66\ \ 6'\ \ 17 = & 9.6075243 \\
\text{A} = 66^\circ\ 6'\ \ 17''. &
\end{array}$$

To find B.

$$\frac{\sin.\ \text{B}}{\sin.\ \text{C}} = \frac{\sin.\ b}{\sin.\ c}$$

Log. sin. B = log. sin. C + log. sin. b + ar. co. log. sin. c

$$\begin{array}{rl}
\text{log. sin. } 81^\circ\ \ 8'\ 11'' = & 9.9947824 \\
\text{log. sin. } 75\ \ 43\ \ 40 = & 9.9863844 \\
\text{ar. co. log. sin. } 87\ \ 28\ \ 35 = & \overline{10}.0004214 \\
\hline
\text{log. sin. } 73\ \ 26\ \ \ 2 = & 9.9815882 \\
\overline{106\ \ 33\ \ 58} &
\end{array}$$

where the supplement of the angle of the tables is chosen that the greater side may be opposite the greater angle.

4. Three divergent lines have the following relations of their parts:

$$A = 110^\circ\, 16', \qquad C = 56^\circ\, 22', \qquad b = 95^\circ\, 36';$$

required the remaining relations.

The parts taken in order are,

$$A,\ c,\ B,\ a,\ C,\ b,\ A;$$

and placing on c two rectangular pyramids, and denoting by S and S′ the segments into which they divide c, we have

$$\overset{+}{\text{cot.}}\ S = \overset{-}{\text{cos.}}\ b\ \overset{-}{\text{tan.}}\ A:$$

$$\begin{array}{lcl}
\text{log. cos. } 84^\circ\ 24 & = & 8.9893737 \\
\text{log. tan. } 69\ \ 44 & = & 10.4326795 \\
\qquad\qquad\ \ \overline{10} & = & \overline{10} \\
\text{log. cot. } 75^\circ\ 11'\ 48'' & = & 9.4220532
\end{array}$$

$$\begin{array}{rcl}
S & = & 75^\circ\ 11'\ 48'' \\
C & = & 56\ \ \ 22\ \ \ 00 \\
S - C & = & 18\ \ \ 49\ \ \ 48'' = S'
\end{array}$$

$$\frac{\text{cos. B}}{\text{cos. A}} = \frac{\text{sin. S}'}{\text{sin. S}}.$$

$$\text{Log. cos. B} = \text{log. cos. A} + \text{log. sin. S}' + \text{ar. co. log. sin. S}$$

$$\begin{array}{rcl}
\text{log. cos. } 69^\circ\ 44'\ 00'' & = & 9.5395653 \\
\text{log. sin. } 18\ \ 49\ \ 48 & = & 9.5088815 \\
\text{ar. co. log. sin. } 75\ \ 11\ \ 48 & = & \overline{1}0.0146595 \\
\text{log. cos. } 83\ \ 21\ \ 34 & = & 9.0631063 \\
B & = & 83^\circ\ 21'\ 34''
\end{array}$$

To find c.

$$\frac{\sin. c}{\sin. b} = \frac{\sin. C}{\sin. B}$$

	log. sin. 84° 24′ 00″ =	9.9979223
	log. sin. 56 22 00 =	9.9204360
ar. co.	log. sin. 83 21 34 =	10.0029235
	log. sin. 56 32 6 =	9.9212818
	c = 56° 32′ 6″.	

To find a.

$$\frac{\sin. a}{\sin. b} = \frac{\sin. A}{\sin. B}$$

	log. sin. 84° 24′ 00″ =	9.9979223
	log. sin. 69 44 00 =	9.9722448
ar. co.	log. sin. 83 21 34 =	10.0029235
	log. sin. 70 2 16 =	9.9730906

As the angle A is greater than B, the side a must be greater than b, or than 95; hence,

	180° 00′ 00″
	70 2 16
a =	109 57 44

5. A solid angle is bounded by the following plane angles:

$$a = 119° 36′, \qquad b = 64° 5′, \qquad c = 79° 56′,$$

required the inclinations of their planes.

Appendix to Section III.

Examples.

$$\begin{array}{rrrl}
 & 119^\circ & 36' & \\
 & 64 & 5 & \\
 & 79 & 56 & \\
\hline
2) & 263 & 37 & \\
\hline
 & 131 & 48 & 30 \\
 & 64 & 5 & \\
\hline
 & 67 & 43 & 30 \ldots \tfrac{1}{2}s - b \\
\hline
 & 79 & 56 & \\
 & 51 & 52 & 30 \ldots \tfrac{1}{2}s - c
\end{array}$$

$$\sin.\tfrac{1}{2}A = \sqrt{\left\{\frac{\sin.(\frac{1}{2}s - b)\,.\,\sin.(\frac{1}{2}s - c)}{\sin.b\,.\,\sin.c}\right\}}$$

$$\log.\sin.\tfrac{1}{2}A = \tfrac{1}{2}\{\log.\sin.(\tfrac{1}{2}s - b) + \log.\sin.(\tfrac{1}{2}s - c) + \text{ar. co.}\log.\sin.b + \text{ar. co.}\log.\sin.c\} + 10$$

$$\begin{array}{rr}
\log.\sin.\ 67\ 43\ 30 = & 9.9663179 \\
\log.\sin.\ 51\ 52\ 30 = & 9.8957902 \\
\text{ar. co. }\log.\sin.\ 64\ \ 5\ \ 0 = & \overline{10}.0460323 \\
\text{ar. co. }\log.\sin.\ 79\ 56\ \ 0 = & \overline{10}.0067379 \\
\hline
2\ ) & \bar{1}{,}9148783 \\
\hline
 & \bar{1}.9574391 \\
 & 10 \\
\hline
65^\circ\ 2'\ 47'' = & 9.9574391 \\
2 & \\
\hline
130\ \ 5\ \ 34 = A &
\end{array}$$

To find B.

$$\frac{\sin.B}{\sin.A} = \frac{\sin.b}{\sin.a}$$

$$\begin{array}{rr}
\log.\sin.\ 49^\circ\ 54'\ 26'' = & 9.8836629 \\
\log.\sin.\ 64\ \ 5\ \ 0 = & 9.9539677 \\
\text{ar. co. }\log.\sin.\ 60\ \ 24\ \ 0 = & \overline{10}.0607329 \\
\hline
\log.\sin.\ 52\ \ 18\ \ 39 = & 9.8983635 \\
52^\circ\ 18'\ 39'' = B &
\end{array}$$

To find C.

$$\frac{\sin. C}{\sin. A} = \frac{\sin. c}{\sin. a}$$

$$\begin{array}{rrrrl}
\log. \sin. & 49° & 54' & 26'' & = 9.8836629 \\
\log. \sin. & 79 & 56 & 00 & = 9.9932621 \\
\text{ar. co. } \log. \sin. & 60 & 24 & 00 & = \underline{10.0607329} \\
& 60 & 1 & 45 & = 9.9376579 \\
& 60° & 1' & 45'' & = C
\end{array}$$

6. The inclinations of three planes which inclose a solid angle, are

$$A = 131° \; 35', \; B = 63° \; 30', \; C = 59° \; 25',$$

required the plane angles.

$$\begin{array}{rr}
& 131\ 35 \\
& 63\ 30 \\
& \underline{59\ 25} \\
2) & \underline{254\ 30} \\
& 127\ 15 = \tfrac{1}{2}\, s \\
& \underline{63\ 30} \\
& 63\ 45 = \tfrac{1}{2}\, S - B \\
& \underline{59\ 25} \\
& 67\ 50 = \tfrac{1}{2}\, S - C
\end{array}$$

$$\cos. \tfrac{1}{2}\, a = \sqrt{\left\{\frac{\cos. (\tfrac{1}{2}\, S - B) \,.\, \cos. (\tfrac{1}{2}\, S - C)}{\sin. B \sin. C}\right\}}$$

$$\log. \cos. \tfrac{1}{2}\, a = \tfrac{1}{2} \{\log. \cos. (\tfrac{1}{2}\, S - B) \log. \cos. (\tfrac{1}{2}\, S - C) + \text{ar. co. } \log. \sin. B + \text{ar. co. } \log. \sin. C\} + 10$$

Appendix to Section III.

Examples.

$$
\begin{array}{rll}
\text{log. cos. } 63^\circ\ 45' & = & 9.6457058 \\
\text{log. cos. } 67\ \ 50 & = & 9.5766892 \\
\text{ar. co. log. sin. } 63\ \ 30 & = & \overline{1}0.0482088 \\
\text{ar. co. log. sin. } 59\ \ 25 & = & \overline{1}0.0650523 \\
 & 2) & \overline{1}.3356561 \\
 & & \overline{1}.6678280 \\
 & & 10 \\
62^\circ\ 15'\ 49\tfrac{1}{2}'' & = & 9.6678280 \\
2 & & \\
124\ \ 31\ \ 39 & = & a
\end{array}
$$

To find *b*.

$$\frac{\text{sin. } b}{\text{sin. } a} = \frac{\text{sin. B}}{\text{sin. A}}$$

$$
\begin{array}{rll}
\text{log. sin. } 55^\circ\ 28'\ 21'' & = & 9.9158504 \\
\text{log. sin. } 63\ \ 30\ \ 00 & = & 9.9517912 \\
\text{ar. co. log. sin. } 48\ \ 25\ \ 00 & = & 10.1261035 \\
80\ \ 17\ \ 57 & = & 9.9937451 \\
b = 80^\circ\ 17'\ 57'' & &
\end{array}
$$

To find *c*.

$$\frac{\text{sin. } c}{\text{sin. } a} = \frac{\text{sin. C}}{\text{sin. A}}$$

$$
\begin{array}{rll}
\text{log. sin. } 55^\circ\ 28'\ 21'' & = & 9.9158504 \\
\text{log. sin. } 59\ \ 25\ \ 00 & = & 9.9349477 \\
\text{ar. co. log. sin. } 48\ \ 25\ \ 00 & = & 10.1261035 \\
\text{log. sin. } 71\ \ 28\ \ 42 & = & 9.9769016 \\
71^\circ\ 28'\ 42'' & = & c
\end{array}
$$

Appendix to Section III.

EXAMPLES FOR PRACTICE.

1. Given $b = 36° 52'$, $A = 24° 17'$, $C = 90°$; required the remaining parts.

Ans. $a = 15° 8' 45''$, $c = 39° 26' 40''$, $B = 70° 47' 28''$.

2. Given $a = 119° 6'$, $B = 44° 37'$, $C = 90°$; required the remaining parts.

Ans. $b = 40° 46'$, $c = 111° 36' 46''$, $A = 109° 58' 24''$.

3. Given $b = 112° 27'$, $B = 117° 30'$, $c = 90$; required the remaining parts.

Ans. $a = 55° 47' 39''$, or $121° 12' 21''$,
$A = 52 \;\; 32 \;\; 9$ or $127 \;\; 27 \;\; 51$,
$C = 73 \;\; 41 \;\; 17$ or $106 \;\; 18 \;\; 43$.

4. Given $A = 104° 19'$, $B = 153 \; 27'$, $c = 90$; required the remaining parts.

Ans. $a = 96° 30' 27''$, $b = 152 \; 43 \; 13$, $C = 102 \; 46 \; 47$.

5. Given $a = 48° 7'$, $c = 59° 19'$, $C = 87° 15'$; required the other parts.

Ans. $b = 43° 18' 36''$, $A = 59 \;\; 50 \; 57$, $B = 52 \;\; 48 \;\; 58$.

6. Given $b = 114° 12'$, $c = 56° 19'$, $B = 121° 50'$; required the other parts.

Ans. $a = 87° 3' 52''$, $C = 50 \; 48 \;\; 39$, $A = 68 \;\; 28 \;\; 7$.

7. Given $B = 46° \; 11' \; 32''$, $C = 133° \; 23' \; 50''$, $a = 73° 19'$; required the other parts.

Ans. $b = 62° 27''$, $c = 116 \;\; 47 \;\; 22$, $A = 51 \;\; 14$.

8. Given $A = 124° 13'$, $C = 49° 7'$, $b = 85° \; 53' \; 27''$; required the other parts.

Ans. $a = 115° 6'$, $c = 55 \;\; 53 \;\; 21$, $B = 65 \;\; 36 \;\; 58$.

Appendix to Section III.
Examples for practice.

9. Given $a = 65° 00' 10''$, $b = 99° 39' 40''$, $c = 38° 31' 20''$; required the other parts.

Ans. $A = 25° 33' 50''$, $B = 152\ 00\ 18$, $C = 17\ 15\ 4$.

10. Given $a = 85° 16'$, $b = 49° 56'$, $c = 112° 19'$; required the other parts.

Ans. $A = 62°, 29' 48''$, $B = 42\ 55\ 54$, $c = 124\ 34\ 40$.

11. Given $A = 71° 42'$, $B = 125° 37'$, $C\ 49° 32'$; required the other parts.

Ans. $a = 95° 56' 10''$, $b = 121\ 36\ 31$, $c = 52\ 50\ 44$.

12. There is a crystal which has one of its obtuse solid angles formed by three planes, mutually inclined at angles of 105° 5′; required the plane angles.

Ans. The three plane angles are each 101° 55′.

SECTION IV.

RELATIONS OF TWO POINTS RESTRICTED TO A GIVEN DISTANCE AND A GIVEN PLANE.

Equation of the circle—a straight line cannot intersect a circle in more than two points—line which is a tangent to a circle—the product of conjugate secants is independent of their direction—the arc of a circle intercepted by two straight lines that diverge from the centre measures their inclination—the sine, cosine, &c. of an angle may be expressed in relations of the arc that measures it—circumference compared with the radius—the inclination of two secants is measured by the difference of the arcs intercepted between them—when the secants intersect in the circumference, their inclination is measured by half the arc intercepted—to find the radius of a circle that shall pass through three given points—to find the radius of a circle that shall be inscribed in a given triangle—differentials of the trigonometrical functions—imaginary formulæ connecting the arc with the trigonometrical functions of it—formulæ of Demoivre—formulæ of Euler—expansions of cos. x^m *and sin.* x^m.

Chap. I. Detailed analysis of the relations of direction; and of the relations peculiar to three, to four, and to a greater number of points.

Art. 160. Equation of the circle.

160. With the view of deviating less widely from the course usually pursued in the analysis of form and magnitude, we shall anticipate, in the two following sections, a subject that belongs to another part of the work, and place among the relations of a definite number of points, other relations common to points limited, indeed, in their position, but infinite in number.

"To discover the properties common to points that have a given distance from a point assigned in space" is the inquiry, the frequent recurrence of which renders this course advisable.

It is usually divided into two problems; and, for the reasons already assigned, we shall not depart from this arrangement, but shall, in the present section, proceed to consider the limited case presented by the first of these divisions, reserving the development of the unrestricted problem to the section which follows.

According to this method of treating the subject, the inquiry on which we are about to enter will have for its object, "to determine the properties common to points lying in a given plane, and at an assigned distance from a point in that plane."

A slight consideration of the analysis required by this question will convince us, that it is sufficient to examine the relations of only two points, one, A, assigned in space, and the other, B, limited by the condition of a given distance.

This circumstance greatly simplifies the problem, and enables us to translate it with facility into the language of algebra.

Thus, denoting the given distance according to the notation of page 109, by the letter a, and regarding this

line as the hypothenuse of a right angled triangle, one of whose sides is a, we have, at once, the equation

$$\frac{x}{a} = \cos. ax \; . \; . \; . \; . \; . \; \alpha$$

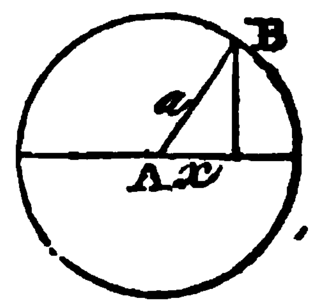
which is common to all the points whose positions and relations are required.

These positions and relations are involved in the equation α, and may be developed from it.

To develope the first, we have merely to cause the direction of x to coincide with some known line in the plane, and to give x, successively, every value from $+ a$ to $- a$.

Pursuing this course, we find the points to be infinite in number, and to form, by their union, the curve described in the first part of the work as the *circle*.

Fig. 208.



The point A is called the *centre* of the circle; the constant distance, a, its *radius*, and the curve itself is also known as the *circumference*, or *periphery*.

A rectilinear figure, that has all its angles in the circumference of a circle, is said to be *inscribed* in it, and the circle is said, conversely, to be *circumscribed* about the figure. When the sides of the latter are equal, it is termed a *regular* figure; Part I., Appendix I.

The equation α is not the only one whereby the positions of the points that constitute a circle can be expressed; many others may readily be deduced.

Thus, denoting by y the side opposite to A, fig. 208, we have,

$$x^2 + y^2 = a^2 \; . \; . \; . \; . \; . \; \beta$$

And, again; assuming three points, A, B and C, and

supposing B to take every position which is consistent with its retaining a given distance from C, we have, for an equation of the circle that has the radius b and centre C,

$$a^2 = b^2 + a'^2 - 2\, ba' \cos.\, ba' \; . \; . \; . \; . \; . \; \gamma$$

And, by a similar analysis, we shall also have,

$$a^2 - 2\, aa' \cos.\, aa' - b^2 + a'^2 = 0 \; . \; . \; . \; . \; . \; \delta$$

After these remarks, we shall proceed to treat of the relations of the circle, under the form of distinct propositions.

161. Prop. 1. A straight line cannot intersect a circle in more than two points.

For the values of a will be given by the two roots of the equation δ.

162. Prop. 2. If a point A is taken without a circle, and lines diverge from the point in the direction of the curve, and these lines are arranged in two divisions, namely, those that fall without the curve, and those which meet it; the line separating these divisions will meet the curve in only one point, and will be at right angles to the radius which passes through the latter.

The roots of the equation δ will cease to be possible, when $a'^2 \sin.^2 aa' > b^2$; which gives for the limit, sin. $aa' = \dfrac{b}{a'}$; or, $(ab) = \frac{1}{4}$. Also, for this value of (ab) the values of a become equal, and the line meets the curve in one point only.

The line we have described is said to be *tangent* to the circle at the point where it meets the latter.

When the line a cuts the circle, it is called a *secant,** and the two values of a are called *conjugate secants.*

163. Prop. 3. The product of conjugate secants is independent of their direction.

For, the values of a are, $a' \cos. aa' + \sqrt{b^2 - a'^2 \sin.^2 aa'}$, and $a' \cos. aa' - \sqrt{b^2 - a'^2 \sin.^2 aa'}$; and, multiplying them together, their product, $a'^2 - b^2$, is independent of the angle (aa').

As a corollary from this proposition it follows, that, if α and β are two conjugate secants, drawn from any point A, and α' and β' two other conjugate secants, drawn from the same point, then, $\alpha\,\beta = \alpha'\,\beta'$; which equation, when α' is equal to β', or the secant α' becomes a tangent, reduces to $\alpha\,\beta = \alpha'^2$.

The equations whence the preceding results were deduced, are independent of the position of the point A, and will equally apply whether that point falls within or without the circle.

In the former case, b is greater than a', and one of the values of a becoming negative, their difference will be a sum.

This difference is a straight line, terminated each way by the circumference.

A line so terminated is called a *chord.*

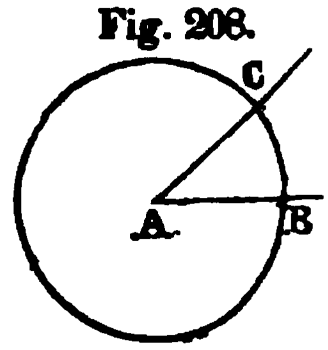
From what is here said, it follows, that when A falls within the circle, the conjugate secants become parts of chords; and hence, if two chords pass through the same

* This line must not be confounded with the ratio of the same name, page 127.

point within a circle, and are divided by it into the parts α and β, α' and β', we shall still have $\alpha\beta = \alpha'\beta'$.

164. Prop. 4. The arc of a circle intercepted by two straight lines that diverge from the centre, measures their inclination.

Fig. 208.



For if the angle at A, added to itself any number of times, art. 42, will produce a space equal to the whole plane space about A, the arc BC will, by the same number of additions, produce the whole circumference. Or, if the angle at A, added to itself any number of times, will produce, art. 7, a space equal to any given number of times the whole plane space about A, the arc BC, repeated the same number of times, will produce the same multiple of the whole circumference. From which it appears, that the ratio of A to the whole space about A, is the same as the ratio of BC to the whole circumference.

This measure is one of great convenience, and was formerly the only one used. We shall frequently adopt it, as an artifice of calculation; but, in doing so, we must be careful, as in former cases, art. 38 and 42, not to confound the measure with the quantity measured. An arc and an angle have little analogy beyond this common ratio which they bear to their natural units; and, if we occasionally make one become the sign of the other, we must yet remember, that error will arise when things so essentially different are confounded. Our elementary writers have been remiss on this point, and have failed to warn the reader of an error that lies immediately in his path.

This want of due notice is essentially felt in those branches of analysis where angles are measured by the ratio which the corresponding arcs of a circle bear to its radius.

The latter becomes, in this case, the unit of lines, and the circumference, as we shall presently show, is measured by twice the endless decimal, 3.14159, &c. already met with in the construction of the trigonometrical tables, and which analysts have, for brevity, denoted by the symbol π.

Now the whole circumference being represented by 2π, an arc that is its mth part will be justly represented by $\frac{2\pi}{m}$; but for this number to represent the corresponding angle, the unit of the latter species of quantity must be measured by the unit of arcs, or by an arc that is equal to the radius; or, in other words, by the 3.14159th part of the whole of plane space.

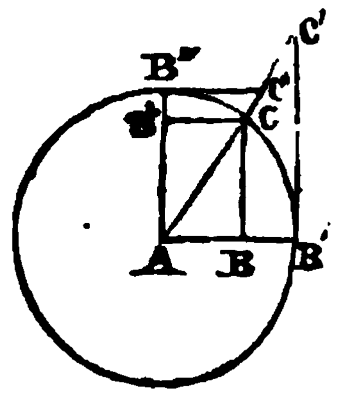
The measures of arcs and of angles are thus shown to be governed by laws that are analogous, without being identical; the laws applying, in one case, to a finite line, the natural unit of which is the radius, and in the other, to an infinite surface, the natural unit of which is the whole of plane space.

165. Prop. 5. Regarding the radius of a circle as the unit of lines, the sine, cosine, &c. of an angle may be expressed by relations of the arc that measures it.

Chap I. Detailed analysis of the relations of direction; and of the relations peculiar to three, to four, and to a greater number of points.

Art. 165. The sine, cosine, &c. of an angle may be expressed in relations of the arc that measures it.

Fig. 209.



Letting fall from C, one extremity of the arc, a perpendicular, CB, upon the radius which passes through the other extremity, we have,

$$\frac{AB}{AC} = AB = \cos. A,$$

$$\frac{BC}{AC} = BC = \sin. A.$$

And erecting at B′, one extremity of the arc, a perpendicular to the radius there, and continuing this perpendicular until it meets a line which is obtained by producing the radius that passes through the other extremity, we have,

$$\frac{BC'}{AB'} = BC' = \tan. A, \frac{AC'}{AB'} = AC' = \sec. A.$$

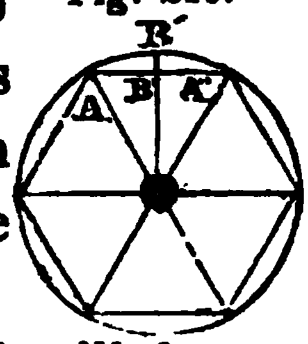
The lines, AB, BC, B′C′, A′C′, which are thus shown to be, respectively, the cosine, sine, tangent and secant of the angle A, are also called the cosine, sine, tangent and secant of the arc B′C, which measures that angle.

Finally, drawing AB‴ at right angles to AB, we shall have $B'''C = \frac{1}{2}\pi - B'C$, and the lines AB″, B″C, B‴C″ and AC″, which are, respectively, the cosine, sine, tangent and secant B‴C, are also the sine, cosine, cotangent and cosecant of B′C.

The fourth part of a circumference is called a *quadrant*, and the arc, B‴C, by which an arc, BC, differs from a quadrant, is called the complement of the latter.

166. Prop. 6. The circumference of a circle which has the diameter unity, is equal to 3.14159, &c.

Fig. 210.



To demonstrate this proposition, take two equal arcs B′A and BA″, on opposite sides of B′. The sines BA and BA″ will fall in one straight line, since the angles at B are right angles.

Now if AB′ is properly assumed, AA″ will be an aliquot part of the circumference; and by continued juxtapositions of AOA″ a regular inscribed rectilinear figure would be formed, having as many sides as there are arcs AA″ in the whole circumference.

Assuming AB′ very small, the number of sides which the figure contains will be proportionally great; and by continually decreasing AB′, we may obtain an inscribed figure of so great a number of sides, that, however large the scale whereon it was drawn, the senses could not distinguish it from the circle,

But again, a side AA″ of the polygon is equal to twice AB, or to twice the sine of AB′, and if AB′ is the mth part of the circumference, the whole periphery of the polygon will be $m \sin. \frac{2\pi}{m}$. And, art. 121, when m increases, the sine divided by the arc continually approaches to a constant limit as its greatest value: wherefore, denoting by the symbol $\measuredangle$ a superior limit, or a number to which the greater value of another number continually approaches, without finally leaving between them a finite difference, we have,

$$\frac{\sin. \frac{2\pi}{m}}{\frac{2\pi}{m}} \measuredangle c, \text{ or } m \sin. \frac{2\pi}{m} \measuredangle 2 c \pi.$$

But when m is great, the periphery of the figure confounds itself with the circumference of the circle, and is

not to be distinguished from it by the senses, however large the scale whereon the polygon is drawn. In other words, the difference between the two figures may be rendered so small that no given multiplication would render it appreciable. This circumference is therefore the limit of the periphery, and we have,

$$2\pi = 2c\pi.$$

or,

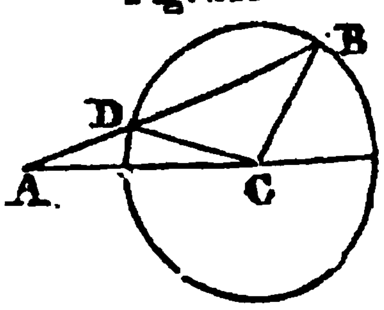
$$c = 1.$$

But c was shown in art. 121, to be 6.28318, &c.; and hence c will have the values unity, or 6.28318, according as the radius, or the circumference, denote the unit of lines; these units, then, must themselves have the same ratios; or the diameter must be to the circumference as 1 to 3.14159, &c.

167. The inclination of two secants will be measured by half the difference of the arcs intercepted between them.

From the theory of triangles we have,

Fig. 211.



$$(b\,a') = (a\,a') + (a\,b);$$

and

$$(a\,b) = (a\,c) = (a\,a') + (c\,a');$$

Whence, by substituting the value of ab,

$$(a\,a') = \tfrac{1}{2}\,(b\,a') - \tfrac{1}{2}\,(c\,a').$$

A similar result might be obtained for a secant that fell below a', and adding the equation thence obtained to that already deduced, we have the result desired.

The equations which we have here deduced, will, by the theory of correlations, be true for every position of A; and consequently, are equally true, whether A falls

Sect. IV. Relations of two points restricted to a given distance and a given plane.

Art. 170. To find the radius of a circle that shall be inscribed in a given triangle.

At the points of contact the sides will be at right angles to the radii. Whence, denoting the triangle by ABC, the centre of the circle sought by D, its radius by r, and the points of contact by E, F and G, we have, analysing by the closed figure AEDG,

$$r = a^v \cos. a^v d^v + r \cos. d'd$$

$$= a^v \sin. aa' - r \cos. aa';$$

or,

$$a^v = \frac{r\,(1 + \cos. aa')}{\sin. aa'};$$

or,

$$a' = r \left\{ \frac{1 + \cos. aa'}{\sin. aa'} + \frac{1 + \cos. ba'}{\sin. ba'} \right\};$$

or,

$$r = \frac{a' \sin. aa' \sin. ba'}{\sin. aa' + \sin. ba' + \sin. ba}$$

$$= \frac{aa' \sin. aa'}{a + b + a'};$$

and substituting for sin. aa' its value obtained in the last problem, we have

$$r = \frac{2 \sqrt{s\,(s - a)\,(s - b)\,(s - a')}}{a + b + a'}.$$

171. In the remainder of the present section we shall regard the reader as acquainted with the differential calculus, and proceed to develope the most useful results that arise from its application to circular arcs and to angles.

The first problem of this kind that presents itself, re-

SECTION V.

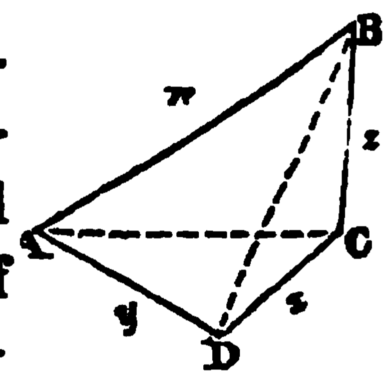
RELATIONS OF TWO POINTS RESTRICTED TO A GIVEN DISTANCE.

Equation of the sphere—the section of a sphere by a plane is a circle—the tangent plane to any point of a spherical surface is at right angles to the radius which passes through that point—if through any point secants are drawn to the sphere, their properties will be the same as those of the secants of the circle—solid angles at the centre of the sphere are measured by the portion of the spherical surface intercepted by them—the solid angle formed by two secant planes is measured by the difference of the spherical surfaces which they intercept—the shortest distance between two points on a sphere is the arc of a great circle intercepted between them—the extremities of the perpendicular drawn from the centre of a sphere to a circle of the latter are every where equally distant from the circle—the angles and sides of a spherical polygon have the same relations as the parts of the solid angle which the polygon subtends at the centre—relations common to all spherical triangles—relations of particular spherical triangles—formulæ for determin-

ing the parts of a spherical triangle when one of those parts is 90—polar spherical triangle—formulæ for determining the parts of any spherical triangle—measure of the surface of a spherical triangle—spherical trigonometry includes plane trigonometry as a particular case—equality by symmetry.

176. The general case of the problem enunciated in art. 160, relating to a point in space, will be most easily resolved by means of the type of solid figures. One of the lines, r, in that type, is distinguished from the rest as being opposite to a right angle in each of the faces to which it belongs. This side we shall hereafter refer to as the "subtend."

Fig. 213.



The sides x, y and z are equally remarkable from their directions, which are mutually rectangular. As the form of the solid is completely determined by the ratios of these four lines, we should be led, in accordance with the method pursued in art. 50, to designate them by peculiar names; but that such a course is not essential to the analysis we use, has been shown in Part II., Chap. I., Sec. IV.; and it may not be amiss to observe, that were the ratios of the solid type named, we could not, in establishing their nomenclature, adhere to the method adopted with regard to the type of plane figures, since in the former, the ratios are not functions* of the solid angle A alone, but functions involving two of the seven angles about that point; or, in other words,

* Any number that depends on and varies with another is called a function of it.

functions involving two of the seven things here mentioned, namely, the solid angle at A, the three solid angles formed by the planes that meet at A, and, finally, the three plane angles at that point. Hence, denoting by S the solid angle at A, and by s the solid angle (A,A′), we might name the ratios as follows:

$$\frac{x}{r} = \text{1st ratio } (S, s),\ \frac{y}{r} = \text{2d ratio } (S, s),\ \frac{z}{r} = \text{3d ratio } (S, s)$$

But as such a notation has not been adopted, we must adhere to the analysis employed in the former part of the work.

According to that analysis we have, representing the angles (aa'') and (A,A′) by θ and ϕ,

$$\frac{y}{r} = \sin.\ \theta \cos.\ \phi,$$

which belongs to all the points that are at the required distance from A.

The analysis by rectangular co-ordinates will also deduce an equation for the surface wherein all the points sought are contained: for since the distance between the given point A, and the point B, which is sought, is expressed, art. 99, by

$$r = \{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2\}$$

we have merely to consider r as a constant, in order to deduce from this expression the position of all the points that shall have the assigned distance from A. To which we might add, that every equation belonging to the circle, belongs equally to the sphere; since, making ϕ constant, the points of the surface that lie in the plane A, will, by the definition a sphere, be at the distance r from A, and consequently form a circle, having the same centre and radius as the sphere itself.

From these equations we might deduce the form of the surface wherein all the points lie that possess the property sought; but, in fact, that surface is more readily understood without the assistance of the equations; for as the angles θ and ϕ may be taken at pleasure, the line r may be directed towards any point in space; and as r is constant, no sufficient reason can be assigned why the form of the surface should differ in one part from its form in another.

The surface sought, which is the *sphere* of Part I., will thus be a closed surface, having all its parts alike.

The given point A is termed the *centre* of the sphere, and the line r its *radius*.

The greatest and least values of x, y and z, will each, by the preceding equations, be $+r$ and $-r$: and the length of a line which passes through the centre and is terminated at either extremity by the surface, will be $2r$: this line is termed a *diameter*.

A solid, bounded by plane surfaces, Part I., App. I., is said to be inscribed in a sphere, when all the angles of the solid lie in the surface of the sphere.

After these general remarks, we shall treat the properties of the figure under the form of distinct propositions.

177. Prop. 1. The section of a sphere by a plane is a circle.

As the equation of the sphere

$$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = r^2$$

was obtained by regarding the co-ordinate planes as rectangular, and without otherwise assigning their positions, we may cause one of these planes, that of the xy's, for

example, to coincide with the plane whose intersection is sought; and thus resolve the question by seeking the intersection of the plane of the xy's with the surface of the sphere.

But the intersection of two surfaces, consisting of points common to both, is found by uniting their equations; and since for all points in the plane of the xy's, and for those points only, z is zero, we may regard $z=0$ as the equation of the plane; and determine the intersection required by substituting this value of z in the equation of the sphere.

Effecting the substitution, we obtain

$$(x-\alpha)^2+(y-\beta)^2=r^2-\gamma^2;$$

which is the equation of a circle having α and β for the co-ordinates of the centre, and $\sqrt{r^2-\gamma^2}$ for the radius.

When $\gamma=0$ the plane passes through the centre of the sphere, and the radius of the circle is r; such a section is called a *great circle* of the sphere.

If from the centre of the sphere a perpendicular is drawn to the plane of the xy's, and any point in the section is united with the foot of this perpendicular, and also with the centre of the sphere; a right angled triangle is formed, having the radius of the sphere for its hypothenuse, and the perpendicular on the plane for one of its sides; and as the remaining side will then be of the same magnitude wherever the point in the intersection is taken, the perpendicular must pass through the centre of the circle which is the intersection of the plane and sphere.

178. Prop. 2. The tangent plane to any point of a

Chap. I. Detailed analysis of the relations of direction; and of the relations peculiar to three, to four, and to a greater number of points.

Art. 178. The tangent plane to a spherical surface is at right angles to the radius. Art. 179. Secants to the sphere and the circle have the same properties. Art. 180. Solid angles at the centre measured by the spherical surface they intercept.

spherical surface is at right angles to the radius which passes through that point.

Assuming in the preceding proposition $\gamma = r$, the intersection of the plane and sphere becomes merely a point; and through this point, by what is there said, the radius, which is at right angles to the plane, will pass. But arranging the planes that can be drawn through the origin into two classes, according as they do or do not meet the sphere; the plane we have described will be a limit separating these classes; and hence, by analogy with the corresponding case in art. 162, will be termed the *tangent plane*.

From what has been said, it is evident that the tangent plane meets the surface only in a point.

179. Prop. 3. If through any point secants are drawn to the sphere, their properties will be the same with those of the secants to a circle.

The section formed by the plane wherein the secants lie having been proven a circle, the proposition may be regarded as demonstrated.

180. Prop. 4. Solid angles at the centre of the sphere are measured by the portion of the spherical surface which they intercept.

The reasoning is the same as that used in art. 164.

181. Prop. 5. The solid angle formed by two secant planes is measured by the difference of the spherical surfaces which they intercept.

The demonstration is the same as that used in art. 167.

182. Prop. 6. The shortest distance between two points on a sphere is the arc of a great circle intercepted between them.

Assume any number of points, A, B, C, D, ... N, to lie in some continuous line upon the surface of the sphere. Through the points A and B, and the centre of the sphere, imagine a plane to be passed; and this done, imagine similar planes to be passed through B and C, C and D, &c.; until planes have been made to pass through the centre, and each pair of contiguous points. We shall by this process obtain the following results.

First, There will be formed on the surface of the sphere a closed polygon, whose sides are the arcs of great circles, *a*, *b*, *c*, &c.

Secondly, There will be formed at the centre of the sphere as many plane angles, measured by these arcs, and enclosing a solid angle that has its vertex at the centre.

But it is a corollary from art. 140, that any one of these angles is less than the sum of the rest; and the arcs measuring the angles, a like result is to be concluded of the arcs also.

Now combining what is here said, it will follow as a first deduction, that in proceeding along the surface of the sphere, from A to N, the route by the arc of a great circle uniting A with N, is shorter than the route by the arcs of great circles which pass through B and C, C and D, and the other intermediate points.

But it will also follow, that, as the points are multiplied, the length of the route increases.

For let us interpose between C and D an additional point D′, that does not lie in the great circle joining C and D. The route from C to D, by way of the arcs that pass through D′ will, by what has already been said, be longer than the route by way of the arc uniting C and D; and since the remaining distances are not altered, the introduction of the additional point must evidently have increased the length of the route.

Now by whatever path on the surface of the sphere we pass from A to N, not only may the points B, C, D, &c. be supposed to lie in that path, but an increase in their number must also cause an increase in the degree of coincidence between the broken line formed of arcs uniting the points, and the line which is the assumed path wherein they lie.

By increasing the number of points, therefore, we both cause the broken line to approach the assumed path, and at the same time cause an increase in the length of the route.

This process may be continued until the broken line and the assumed path do not differ by any assignable quantity; or, in other words, until they agree so nearly, that, neither could the senses perceive any difference between them, nor could such difference be perceived by the senses were they improved to any assignable degree of perfection.

The excess of the route by the broken line above the route by the great circle, is thus shown to be a finite quantity, and the difference of the routes by the broken line and the assumed path not being a finite quantity, the

route by the latter must exceed the route by the arc of the great circle; and the length of this arc must be less than the length of any other line that can be drawn on the surface of the sphere between the points A and N that lie on that surface.*

The distance of two points on the surface of a sphere is measured by the arc of the great circle that lies between them.

183. Prop. 7. Drawing from the centre a perpendicular to the plane of a circle of the sphere, and producing this perpendicular either way to the surface, the circle will be everywhere equally distant from the extremities of the perpendicular.

The ratio between the radius of the sphere and the radius of the circle is the sine of the arc intercepted between an extremity of the diameter and any point in the circle; and hence, wherever this point is chosen, the arc in question will be the same.

These equidistant points are called *poles* of the circle, and the poles of a circle are, therefore, the extremities of the diameter which is perpendicular to its plane.

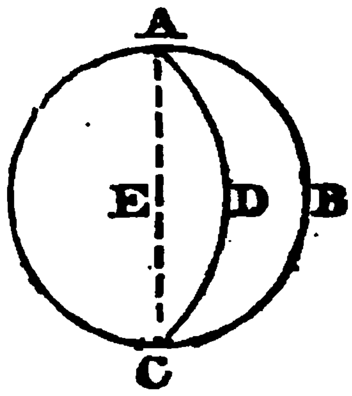
184. When a solid angle is placed at the centre of a sphere, the plane angles that bound it intersect the surface in so many arcs of great circles, which, taken together, form a *spherical polygon*.

The arcs are the sides of this polygon, but its angles, inasmuch as they are formed upon a curved surface, will

* See note 8.

be different from any that have hitherto occupied our attention. Before, therefore, we proceed further in this inquiry, it will be necessary to investigate the nature of such angles, and the standard whereby they are measured: a moment's attention, however, will suffice for this purpose, and enable us to discover the natural unit of such quantities, and the relations they have to angles of a class already considered.

Fig. 214.



For our idea of an angle being the space included between its sides, we may take the spherical area ABCD as the angle at A; and with this definition the natural unit of spherical angles will be the whole surface of the sphere; a unit which varies with the magnitude of the latter.

A convenient measure of another class may also be obtained; for since the area ABCD measures the solid angle included between the planes ADCE and ABCE, the solid may be taken as the measure of the spherical angle, and either of these quantities will then be denoted by the same number. With such a measure, the solid angle and the spherical polygon will have all their corresponding parts denoted by the same numbers: the inclinations, for example, of the planes that form the solid angle, will be denoted by the same numbers as the angles of the spherical polygon, and the plane angles of the former will be denoted by the same numbers as the sides of the latter. And as the algebraic expressions for these two varieties of form will thus be identical, the rules and formulæ that we have deduced in Sec. III. of the present Part, for

the relations of a solid angle, will equally apply to the relations of its corresponding spherical polygon. Whence the following propositions :

RELATIONS COMMON TO ALL SPHERICAL TRIANGLES.

185—1. The sum of any two sides of a spherical triangle is greater than the third side ; art. 157—1.

2. The sum of the three sides is less than unity ; art. 157—2.

3. The sum of the three spherical angles is greater than $\frac{1}{2}$, and less than $\frac{3}{2}$; art. 157—3.

4. The sum of any two of the spherical angles is greater than the supplement of the third angle ; art. 157—4.

5. The sum of any two sides must be of the same kind, with respect to 180, as the sum of the opposite angles ; art. 157—5.

PARTICULAR RELATIONS OF SPHERICAL TRIANGLES.

186—1. An equilateral spherical triangle is also equiangular ; art. 158—1.

2. An isosceles spherical triangle has the angles opposite to the equal sides equal, and the converse ; art. 158—2.

187. Formulæ for determining the parts of a spherical triangle, when one of those parts is 90.—(*See art.* 149.)

Chap. I. Detailed analysis of the relations of direction; and of the relations peculiar to three, to four, and to a greater number of points.

Art. 188. Polar spherical triangles. Art. 189. Formulæ for determining the parts of any spherical triangle.

188. Polar spherical triangles.

If a, b and c are the sides, and A, B and C the opposite angles of a spherical triangle, a second triangle exists, such, that

$$a = 180 - A', \; b = 180 - B', \; c = 180 - C'$$
$$A = 180 - a', \; B = 180 - b', \; C = 180 - c'$$

where the notation is so chosen, that a side a', of the second triangle, is opposite to the angle A, of the same name, in the first; art. 148.

Any one of the angular points of either of these triangles, is the pole of the opposite side of the other triangle; on which account the triangles are said to be polar to each other.

189. Formulæ for determining the parts of any spherical triangle.

These formulæ have already been deduced in the Third Section; and in order to render them, as they are there given, applicable to spherical triangles, we have merely to regard the small letters as denoting the sides, and the great letters of the same name as denoting the opposite angles of the triangle.

190. The portions of the spherical surface intercepted by a solid angle at the centre being the same part of the whole surface as the solid angle is of the whole of space, the area of a spherical triangle, when the surface of the sphere is unity, will be obtained by the same rule as the solid angle to which it is opposite. Hence, if S is the

surface, and A, B, C are the angles of the triangle, we have, art. 67,

$$S = \tfrac{1}{2}\{A + B + C\} - \tfrac{1}{4}.$$

191. Before leaving this subject, it will not be improper to notice that spherical trigonometry includes plane trigonometry as a particular case. The triangles measured by the surveyor, on a level field, although traced by the assistance of straight lines, are, in fact, spherical triangles; and from this instance alone it will be seen, that if the radius of the sphere increases to infinity, whilst the size of the figure remains the same, the latter degenerates into a plane triangle.

The formulæ of spherical trigonometry must, therefore, necessarily have a great analogy to those of plane; and when the radius of the sphere is made infinite, and the usual rules for combining infinitely small quantities are observed, the relations of the former species of quantity must become identical with those used in the latter. The reduction is effected by considering the sine and tangent as agreeing, when very small, with the arc; whilst the cosine and secant, the common limit of which, we recollect, is unity, are represented by unity minus the square of an infinitely small quantity.

192. In concluding this section we may also notice a relation of spherical polygons, and of solid figures in general, that ought, perhaps, to have been more fully developed whilst treating of the type of solid figures. This relationship is termed by Legendre, who seems first to have noticed it, an *equality by symmetry;* and will be readily understood by referring to the idea which we have attached to the term equality: that term, it will

be recollected, implies such an identity between the parts of two figures, and the disposition of those parts, that one of the figures could, by superposition, be made to cover the other, or to occupy, in every respect, the space which the latter occupies. When two plane figures are equal, this capacity of being superposed always exists, and is, in fact, the only idea we form of this equality: but when the figures are solid, or are described upon a surface that is not plane, an identity between all the parts does not imply a capacity of superposition; since the parts may be so disposed, that in whatever way we turn the figures, their equal lines and angles may still be differently arranged.

A clear idea of this fact may be obtained from considering the two plane triangles, ABC, BCD, whose corresponding sides and angles are equal. These triangles may be superposed; but it will be observed that in turning the triangle BDC about BC, with the view of superposing it upon ABC, we cause, if the triangles are material, the posterior surface of BCD to change its position, and to turn towards the direction previously faced by the anterior surface. Now this operation is not possible when the figures are two rectangular pyramids, ABEC, BEDA, having their corresponding parts equal. The faces of the two solids may be applied successively, but the direction in which the solid BEDA must be turned, to apply the face ADE to ACE, is not the same as the direction wherein it must be turned to apply the ends BDA, BCA; and the superposition is, therefore, impossible.

Fig. 215.

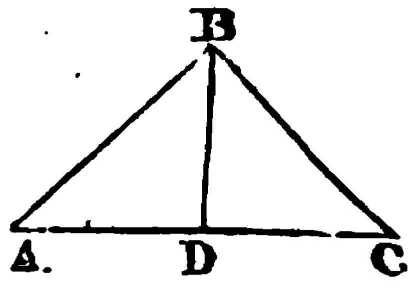
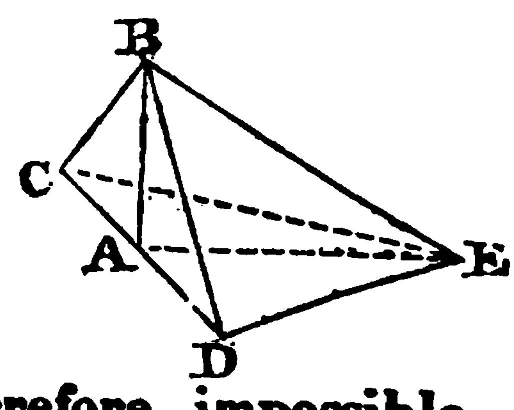


Fig. 216.



This circumstance requires particular attention in synthetic geometry, where the neglect of it has frequently vitiated the demonstrations, but the division of equality into an absolute equality, and an equality by symmetry, although the distinction should never be lost sight of, will not affect the investigation of Part II., Sec. IV.

SECTION VI.

SYSTEMS OF PRIMORDIAL ELEMENTS THAT HAVE REFERENCE TO THE SPHERE; ADDITIONAL THEOREMS FOR TRANSFORMING CO-ORDINATES.

The position of a point on the surface of a sphere is referred to the centre, to a great circle, and to a point arbitrarily chosen in the latter—of primary and secondary circles—distance of two points in terms of their spherical co-ordinates—relative directions of points expressed in terms of their spherical co-ordinates—transformation of spherical co-ordinates—transformation of polar systems—equations of transformation used by Euler.

193. The theory of lines that diverge from a common centre is of such great importance, that although two sections have already been devoted to it, we shall resume the subject, and trace its application to the theory of primordial elements, and thence to the important problems which occupy the attention of the geographer and

Fig. 218.

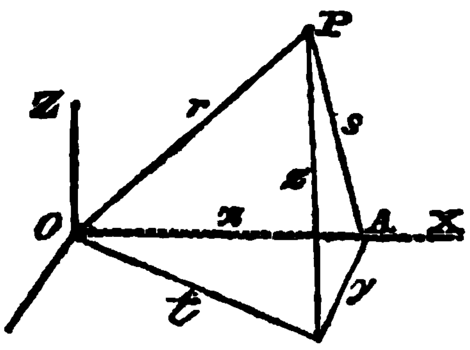
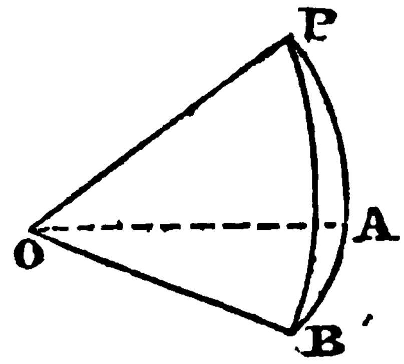


Fig. 219.

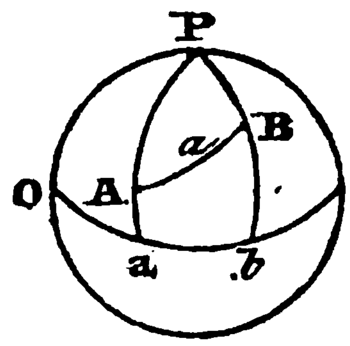


four cases alluded to, is the subtense OP; but the angles whereby the position of this subtense is assigned, are the angles $(x\,t)$ and $(t\,r)$, or, in fig. 219, the arcs AB and BP.

After thus fully explaining the method of assigning the position of a point on the surface of a sphere, or, if the radius of the latter may be assumed at pleasure, the position of a point in space; it remains to derive the formulæ which express the distances of points, and the angles formed by their directions, and those other formulæ by which a transposition of systems is effected.

195. The first of these problems may be resolved by means of the differences and the complements of the coordinates; for assuming A and B as the given points, O as the origin, P as the pole of the primary, and

Fig. 220.



$$Oa = x, \; Ob = x',$$
$$aA = y, \; bB = y',$$

as the circular co-ordinates; we shall have, $PA = \frac{1}{4} - y$, $PB = \frac{1}{4} - y'$, and the angle $APB = x' - x$; whence, denoting the distance sought, AB by d, there results

$$\cos. d = \sin. y \sin. y' + \cos. y \cos. y' \cos. (x' - x).$$

As, however, this formula is not adapted to logarithms, its place is conveniently supplied by either of the two known methods of solving a spherical triangle from the sides and the included angle as data.

The great circle acting, with regard to spherical sur-

for the particular case wherein the origin is the intersection of the original and the transformed primordial circles, it is to that case alone that we shall confine our investigation.

Assuming, then, A to be the given point, P and P′ the poles of the primordial circles, and m the secondary passing through both these poles, we have,

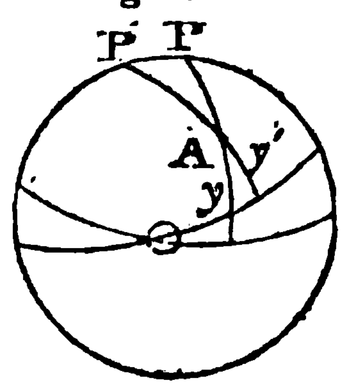
$$P P' = (x\, x'),^{*}\ (m\, y) = (\tfrac{1}{4} + x)\text{; and from } A\, P\, P'$$

$$\sin.\, y' = \cos.\, x\, x' \sin.\, y \dotminus \sin.\, x\, x' \cos.\, y \sin.\, x,$$

$$\cos.\, x' = \cos.\, x \frac{\sin.\, y}{\sin.\, y'}$$

The last of these two formulæ gives x' in terms of y', but y' having been previously determined, the transformation is sufficient for the purposes to which it is usually applied: it should be remarked, however, that in practice y' is not commonly determined from the value here given, but is obtained by solving the triangle APP′ according to either of the methods already deduced, and which admit the ready application of logarithms.

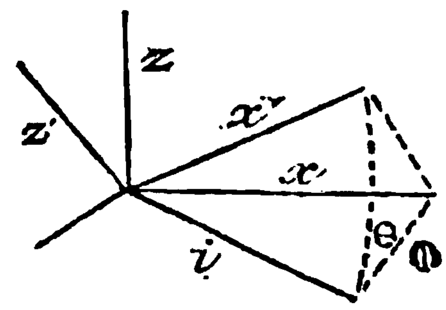
Fig. 221.



198. With these remarks we might terminate our inquiry into the method of measuring angles by the arcs they subtend on the surface of a sphere; but as some of the most eminent modern analysts have used this method for another purpose, and endeavoured to avoid, by its assistance, the eliminations incident to the formulæ of transformation given in page 223, we have thought it best to reserve to the present occasion all that we have to say concerning the transformation of polar systems.

* This is the angle between the two primary circles.

Fig. 222.



These three lines, the axe of the x's, the axe of the x''s, and the intersections of the planes, diverge from the origin, and form there angles whose relations are given by preceding formulæ.

Denoting the angle $i\,x'$ by ψ, the angle $i\,x$ by ϕ, and, finally, denoting by θ the inclination of the planes, the intersection of which is i, we have

$$\cos.\, xx' = \cos.\, \psi \cos.\, \phi + \sin.\, \psi \sin.\, \phi \cos.\, \theta.$$

And since $(i\,y') = (\frac{\pi}{2} + i\,x') = \frac{\pi}{2} + \psi$, the cosine of $x\,y'$ will be obtained by merely substituting, in the preceding expression, $\frac{\pi}{2} + \psi$ for ψ.

The cosine of $y\,y'$ will be found from cos. $x\,x'$ by substituting $\frac{\pi}{2} + \psi$ for ψ, and $\frac{\pi}{2} + \phi$ for ϕ; whilst for cosine $x'\,y$ we merely make the last of these two changes.

The angles which z' forms with x, y and z, have yet to be found, but the investigation is readily effected: one of the angles is, indeed, immediately perceived; for as lines perpendicular to planes are inclined at the same angles (art. 145),* we have,

$$zz' = \theta.$$

To discover the value of $(z\,x')$ we may observe, that, since the plane which passes through z and z' is perpendicular to i, its inclination with the plane zx will be measured by $\frac{\pi}{2} + \phi$; and as we thus know the plane angles $(zx) = \frac{\pi}{2}$, $(zz') = \theta$, and the inclination of their planes, which is equal to $\frac{\pi}{2} + \phi$, the third plane angle

* The proposition referred to in art. 145 will be demonstrated in the next Section.

SECTION VII.

RELATIONS OF ANY NUMBER OF DIVERGENT LINES.

Relations of divergent lines agree with those of spherical polygons—equations relative to the inclinations of divergent lines obtained from the sides of closed figures, by equating with zero the common denominator found in the values of the latter—in a plane, lines equally inclined to other lines form the same angle as the latter—applies also to planes which have a common intersection—perpendiculars to the sides of a closed figure, or to planes that include a solid angle, have for their inclinations the supplements of the internal angles of the latter—sum of the interior angles of a polygon—other relations of divergent lines—data required to determine the relations of divergent lines.

200. By what we have seen in Section V., the relations of any number of divergent lines reduce themselves to relations of the sides and angles of a spherical polygon; and as the method of resolving such polygons by dividing them into spherical triangles is sufficiently ob-

vious, it will be unnecessary to say more concerning the relations of divergent lines than will suffice to render the analysis of them direct and convenient.

201. The method used in art. 139, to obtain an equation of condition between three divergent lines that lie in one plane, will apply to any number of divergent lines; and will, when the number exceeds three, apply equally, whether the lines do or do not lie in one plane, since in such cases it is always possible to form a closed figure having its sides parallel to the given directions.

And the inclinations of the planes that pass through the divergent lines will also be subject to equations of a similar kind, and that may be obtained by the proposition used in the latter part of art. 145. According to which, if a', a'' and a''' are divergent lines, and p', p'', p''' are perpendiculars to the planes passing through these lines, planes that we may denote by $a'a''$, $a'a'''$, $a''a'''$, the inclinations of these planes will be the same as the inclinations of their perpendiculars.

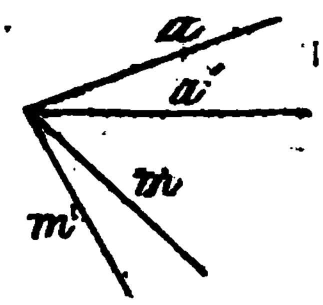
In using this proposition, however, we must remark, that if the angles formed by the planes are estimated as the internal angles of the spherical polygon above alluded to, the supplements of the angles formed by the perpendiculars, and not the angles themselves, will be equal to the inclinations of the planes.

202. This useful theorem has not not yet been demonstrated; it was referred to in art. 145 as a subsequent proposition, and, in fact, forms part of the theory we are

now discussing, and can in no division of our subject be treated more conveniently than in the present.

To consider the subject with the detail that it deserves, let us commence with the simple case of two divergent lines, m and m', that form equal angles θ, with given divergent lines a and a'. By the enunciation we have

Fig. 223.



$$(am) = \theta, \ (am') = (aa' + \theta),$$

and subtracting the first of these equations from the second, there arises the result,

$$(mm') = (aa');$$

and as equations of the same kind will be true for any other lines similarly situated, we may enunciate the proposition as follows: "If in a plane there are two systems of right lines, a, a', a'', m, m', m'', &c. so related that any line $a'^{\cdots n}$ being taken in the first system, a line $m''^{\cdots n}$ can be found in the second, such, that $(a''^{\cdots n} m''^{\cdots n})$ shall be a constant angle θ; then will the inclinations of the lines in the first system be equal to the inclinations of the corresponding lines in the second.

The most useful cases are when θ is equal to 0, or to $\frac{1}{4}$.

203. If the first system of lines do not lie in one plane, the condition which connects the second system with the first will be insufficient to define the lines of that system; and the proposition under the form above given will be inapplicable. The modification that it requires is readily deduced, but the limits prescribed to the present work will not admit of extensive details, and the case wherein the first system is composed of planes, and

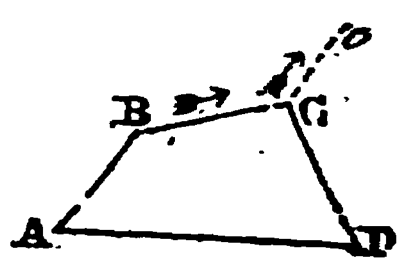
the second of lines perpendicular to them, is a problem of greater interest.

This problem, also, subdivides itself into two cases, according as the planes have or have not, a common intersection. In the first case, the system of perpendiculars lies in a plane at right angles to the common intersection, and as the lines, the inclinations of which measure the inclinations of the planes, are also perpendicular to the common intersection, the problem is reduced to that already considered, or, in other words, to the relations between a system of lines that lie in one plane, and their perpendiculars.

204. When the planes have not a common intersection, the principle on which their inclinations are measured must be particularly described, and the theorem will vary with the different methods used for this purpose.

The rule for determining the sequence of planes adopted in art. 44,* regards them as estimated in the same order as the sides of a closed figure, and hence the theorem we have in view will be best illustrated by considering the relations of a system of lines forming known angles with the sides of a closed figure.

Fig. 224.

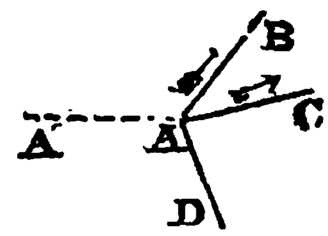


Assuming ABCD, fig. 224, for this last, and drawing from any point A, lines AB, AC, AD, AA′, respectively, parallel to the sides that terminate in the points B, C, D and A′, we readily perceive that the angles of figure 225, estimated according to the rules usually employed, for an-

* See fig. 202, art. 145, and note 7.

gles about a point, are the supplements of the angles of figure 224, estimated according to the usual rule for closed figures.

Fig. 225.



The latter rule, in fact, art. 111, teaches us to estimate the angles as formed by sides passed over in the same directions as the sides of the closed figure, whilst, according to the former rule, the direction of one of the sides of each angle is inverted.

To render this truth more evident, let us select from fig. 224, any two sides AB, BC; and from fig. 225, the lines which are respectively parallel to these sides. The inclination in the former case will be measured by BC c, fig. 224, and in the latter by BAC, fig. 225; and as the sides of BC c have the same directions as the sides of the polygon, whilst a side of BAC has its direction inverted, the angles, according to the reasoning used in art. 111, must be supplementary to each other.

The remainder of the proposition will evidently follow from what has already been said: for a system of lines related to the sides of the polygon by the condition that each line shall make a constant angle with the corresponding side of that figure, will be related by the same condition to the system of parallels, fig. 225, and by art. 202, will have its sides inclined at the same angles as the corresponding sides of these parallels, or at angles supplementary to the inclinations of the sides of the closed figure.

The rule thus obtained will guard us against the mistakes that we might otherwise have committed in estimating the directions of lines perpendicular, or parallel to other given lines. The inclination of the perpendi-

Figs. 226.

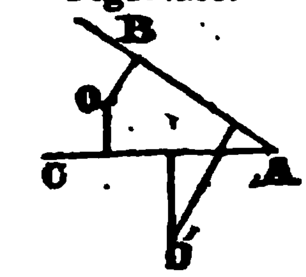
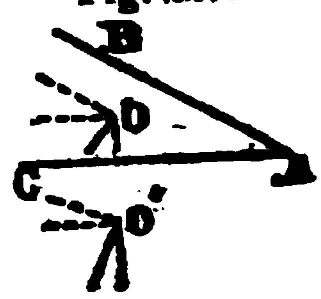


Fig. 227.



culars drawn from O, for example, seems to differ from the inclination of the perpendiculars drawn from O′, fig. 226; but if we previously draw from O and from O′, fig. 227, lines parallel to AB and AC, the angle formed by the perpendiculars will be found the same at either of these points.

The rule will still hold, when, in place of a closed figure and the lines perpendiculars to its sides, we have given a system of planes enclosing a solid angle, and a system of lines perpendicular to these planes; or, in other words, with such data, the inclinations of the planes estimated by art. 44 will be supplementary to the inclinations of the perpendiculars; for as this proposition has been proved for all planes that have a common intersection, it must be true of each pair of planes separately, and, therefore, for the whole system.

Were the point whence the perpendiculars are drawn always situated within the solid angle, and were the latter also of the kind usually implied by that term, the simple proof deduced in art. 145 from considering figure 201, would supersede the details here given; but as the point O′ might have fallen either within or without the figure, and as the solid angle at O admits the extreme and ambiguous cases observed in treating closed figures, art. 40, it became necessary to establish a rule of greater generality, and which should estimate the angles formed by perpendiculars and parallels upon the principles used in estimating the angles of other divergent lines.

205. The theorem which asserts the inclinations of lines diverging from a point, and in the directions of the sides of a closed figure, to be the supplements of the internal angles of that figure, may also be usefully employed in establishing other propositions. Assuming, for example, the internal angles of a figure of n sides to be θ, θ', θ'', &c.; the corresponding angles of the divergent lines will be $\frac{1}{2} - \theta$, $\frac{1}{2} - \theta'$, $\frac{1}{2} - \theta''$, &c. And as the sum of the angles of this system makes up the whole space about the point whence the lines diverge, we have

$$\tfrac{1}{2}\, n - (\theta + \theta' + \theta'' \ \&c.) = 1,$$

or,

$$\theta + \theta' + \theta'' \ \&c. = \tfrac{1}{2}\, n - 1,$$

a result usually expressed as the following proposition: "The interior angles of any closed plane figure are together equal to twice as many right angles as the figure has sides, minus four right angles."

206. And as a second application of the principle in question, or, which amounts to the same, of the dependence between the relations of divergent lines and of closed figures, let us assume the point whence the lines diverge, as the centre of a circle the radius of which is unity, and unite the points wherein the circumference of this circle cuts the divergent lines. A polygon will thus be formed the sides of which have the following simple relations to the divergent lines.

The latter will form as many isosceles triangles as the polygon has sides.

Calling θ the angle at the vertex, and ϕ an angle at the base of one of these triangles, we shall have

$$2\phi + \theta = \tfrac{1}{2};$$

or,

$$\phi = \tfrac{1}{4} - \tfrac{1}{2}\theta.$$

And hence, putting ψ, ψ', ψ'', &c. for the interior angles of the figure, there results,

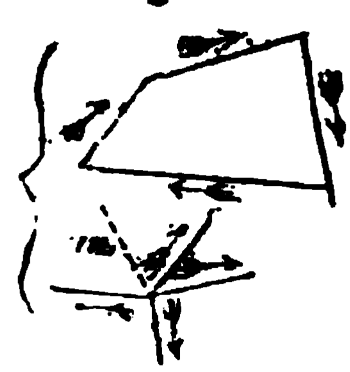
$$\begin{aligned} \psi &= \tfrac{1}{2} - \tfrac{1}{2}(\theta + \theta') \\ \psi' &= \tfrac{1}{2} - \tfrac{1}{2}(\theta' + \theta'') \\ &\text{\&c.} \;\ldots\ldots\; \text{\&c.} \end{aligned}$$

And putting s, s', s'', &c. for the sides, we have, also,

$$\begin{aligned} s &= 2 \sin. \tfrac{1}{2}\theta, \\ s' &= 2 \sin. \tfrac{1}{2}\theta, \\ &\text{\&c.} \;..\; \text{\&c.} \end{aligned}$$

Now in any plane closed figure, if lines are drawn parallel to the sides, the inclination of these divergent lines, by art. 204, will be the supplements of the interior angles; and hence, if we assume a line m, inclined to the first of these lines at the angle ε, and reckon, as above, the interior angles of the figure to be ψ, ψ', ψ'', &c., the inclinations of the several divergent lines, and consequently of the sides of the figure, to m, will be

Fig. 228.



$$\varepsilon,\ \varepsilon + \tfrac{1}{2} - \psi,\ \varepsilon + 1 - (\psi + \psi'),\ \varepsilon + \tfrac{3}{2} - (\psi + \psi' + \psi'') - \text{\&c.}$$

whence, substituting the values of s, s', s'', &c., $(m\,s)$, $(m\,s')$, $(m\,s'')$, &c. in the third form of theorem 9, art. 112, we have

$$0 = \sin. \tfrac{1}{2}\theta \cos. \{\varepsilon\} + \sin. \tfrac{1}{2}\theta' \cos. \{\varepsilon + \tfrac{1}{2}(\theta + \theta')\} + \sin. \tfrac{1}{2}\theta'' \cos. \{\varepsilon + (\theta + 2\theta' + \theta'')\} + \sin. \tfrac{1}{2}\theta''' \cos. \{\varepsilon + (\theta + 2\theta' + 2\theta'' + \theta''')\} + \text{\&c.} \ldots\ldots \quad \alpha$$

where ε may be any angle whatever, and θ, θ', &c. any angles whose sum is equal to unity.

the first class wherein the data exceed three; any one of the second wherein the data exceed five; nor any one of the last class wherein the data exceed seven. Whence the given things are such as suffice to determine the figure.

their relations $\frac{n.(n-1)}{2}$ distances, $\frac{n.(n-1)\{n.(n-1)-2\}}{(1\cdot 2)^2\cdot 2}$ plane angles, $\frac{n.(n-1)(n-2)}{1\cdot 2\cdot 3}$ planes, and, finally, $\frac{n.(n-1)(n-2)\{n.(n-1)(n-2)-6\}}{(1\cdot 2\cdot 3)^2\cdot 2}$ inclinations of planes; beside those other solid angles that are obtained by comparing the planes three or more at a time.

209. The angles included in the analysis of any number of points, involving the same relations as the angles formed by divergent lines, are subject to conditions not dependent on the magnitude, and in some cases, not upon the ratios of the lines whereby the points are connected.

The first of these results, amounting merely to the independence of form and magnitude, will not require illustration; but as an example of the second, we may refer to the proposition which asserts the interior angles of a plane figure to be equivalent to twice as many right angles as the figure has sides, minus two, art. 205.

And as conditions exist involving merely the directions of the points, so also we have conditions that apply merely to their distances. But this is not the case with the distances that form a closed figure; the sides of the latter are subject to no conditions wherein the angles have not a part, or, in other words, we cannot, whilst the angles are indeterminate, form an equation merely from the "sides" of a closed figure. In a triangle, for example, if the angles are restricted to particular values, an equation may exist that appears only to involve the

which amounts to the same, the primordial elements may be assumed as agreeing with parts of the figure investigated.

Proceeding on these principles, we may reason as follows: $3n - 9$ equations will suffice to render the co-ordinates of *all* the points dependent on those of three arbitrarily chosen, on the co-ordinates, for example, of A, of B and of C. But these last may be any co-ordinates that permit the mutual relations of A, B and C; and since the relations of these points are determined when three of them are known, the co-ordinates of A, B and C will be restricted by a like number of equations; which, added to the $3n - 9$ above mentioned, will give $3(n - 2)$; and as these equations are independent, they will require a like number of data to assign the relations of the points.

The method of distinguishing the data that are superfluous from those which suffice for the solution of the problem, agrees with that of art. 207: the analysis must be conducted by figures of three, of four, of five, and of a greater number of sides; and if on examining these an excess of data is not found in any of them, we may conclude the parts assigned to be independent, and such as suffice to determine the remaining relations.

212. The great extent of the subject, and the details necessary to each problem, will not allow us to illustrate by many examples the theory discussed in the present section; we may render, however, the process sufficiently clear by selecting a few problems requiring all, or most of the rules laid down.

Let it be required, for example, to analyse the relations of four points, with the view of determining the

inclinations of those lines that are opposed, or which do not meet.

As n in this case is 4, the data must be six in number, and must be such that, analysing the points by triangles, no one of the latter shall contain more than three of the given relations.

Denoting the points by A, B, C and D, the distances, according to the notation of page 109, will be a, b, c and a''; and the rules of art. 114 will give, for the first closed figure of four sides ABCD, the equations

$$\begin{aligned}
a &= b \cos. ab + c \cos. ac + a'' \cos. aa'' \\
b &= c \cos. bc + a'' \cos. ba'' + a \cos. ba \\
c &= a'' \cos. ca'' + a \cos. ca + b \cos. cb \\
a'' &= a \cos. a''a + b \cos. a''b + c \cos. a''c\,;
\end{aligned}$$

And multiplying each of these equations by its left hand number, adding the second and fourth, and, finally, subtracting the first of the sums from the second, we obtain

$$a^2 + c^2 - b^2 - a''^2 = 2\,ac \cos. ac - 2\,a''b \cos. a''b.$$

Now the combinations 2 and 3, art. 114—2, page 263,* give the following arrangements of the sides of the quadrilaterals used in this analysis:

$$\begin{array}{cccc}
a & b & c & a'' \\
a' & \overline{b} & b' & a'' \\
a & b' & \overline{c} & \overline{a'}
\end{array}$$

And substituting for a, b, c, &c. in the last equation the values here given, and observing to change the signs of the cosines for those letters that have minus written above them, we deduce two other equations, which, to-

* Combination 3 of the article referred to is, by an inadvertency, merely the preceding combination written backward: it should be as given above.

gether with the equation already deduced, form the system

$$a^2 + c^2 - b^2 - a''^2 = 2ac \cos. ac - 2a''b \cos. a''b$$
$$b^2 + a''^2 - a'^2 - b'^2 = -2a'b' \cos. a'b' - 2a''b \cos. a''b$$
$$b'^2 + a'^2 - a^2 - c^2 = 2ac \cos. ac - 2a'b' \cos. a'b'$$

And from these again, by changing the signs of a single equation, and adding the three together, we successively deduce

$$\cos. ac = \frac{a'^2 - b^2 + b'^2 - a''^2}{2ac}$$

$$-\cos. a''b = \frac{a^2 - b'^2 + c^2 - a'^2}{2a''b}$$

$$-\cos. a'b' = \frac{a''^2 - c^2 + b^2 - a^2}{2a'b'}$$

formulæ that express the inclination of those sides of a quadrilateral that are opposed, and which, unless the parts lie in a plane, do not meet.

The want of homogeneity apparent in the left hand members of these equations results from the angles being estimated according to the rule used for a single closed figure of four sides; whereas the three pair of sides that are opposed, should be regarded as belonging respectively to each of the three quadrilaterals used in the analysis. With this restriction, the equations become

$$\cos. ac = \frac{a'^2 - b^2 + b'^2 - a''^2}{2\,ac}$$

$$\cos. a''b = \frac{a^2 - b'^2 + c^2 - a'^2}{2a''b}$$

$$\cos. a'b' = \frac{a''^2 - c^2 + b^2 - a^2}{2a'b'}$$

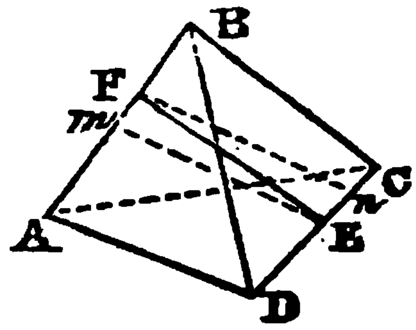
By adding them together we obtain an equation of condition remarkable for the simplicity of its form; and

from its containing only the relations of those sides that do not meet, the equation is

$$ac \cos. ac + a''b \cos. a''b + a'b' \cos. a'b' = 0.$$

As a second example of the analysis of four points, let it be required to determine the shortest distance between the opposite sides, the inclination of which has just been found.

Fig. 231.



The shortest distance from a point to a straight line is, evidently, the perpendicular let fall upon the latter; since, by art. 50—1, the hypothenuse of a right angled triangle is longer than either of the sides.

The shortest distance between two straight lines will, therefore, be perpendicular to both.

For, taking AB and CD as the lines, and FE as the shortest distance between them; if FE were not at right angles to CD, we might find a perpendicular F n less than FE; and, in the same manner, if EF were not perpendicular to AB, we might find a perpendicular E m less than EF; and as both these suppositions are inconsistent with the hypothesis, we conclude EF to be at right angles to each of the lines.

The relations of the points will, therefore, admit of being resolved into four quadrilaterals, each containing two right angles. The quadrilateral ADEFA will suffice, however, for the analysis we propose. We obtain from it the equations

$$\begin{aligned}
e &= a'' \cos. a''e \\
a^{\text{v}} &= a'' \cos. a''a^{\text{v}} + d \cos. da^{\text{v}} \\
a'' &= d \cos. da'' + e \cos. ea'' + a^{\text{v}} \cos. a^{\text{v}}a'' \\
d &= a^{\text{v}} \cos. a''d + a'' \cos. a''d.
\end{aligned}$$

And since a^{v} and d have the same directions as a and c, these equations change into

$$e = a'' \cos. a''e$$
$$a^{\text{v}} = a'' \cos. a''a + d \cos. ca$$
$$a'' = d \cos. ca + e \cos. ea'' + a^{\text{v}} \cos. aa''$$
$$d = a^{\text{v}} \cos. ac + a'' \cos. a''c.$$

The distance a'' and the angles $a''a$, $a''c$ and ac, are data sufficient to determine the lines AB, DC, and consequently to assign the shortest distance between them.

Eliminating from the second and fourth of the equations those quantities not found in the data here mentioned, we obtain

$$a^{\text{iv}} = \frac{a'' (\cos. a''a + \cos. a''c \cos. ac)}{\sin. ac^2}$$

$$d = \frac{a'' (\cos. a''c + \cos. a''a \cos. ac)}{\sin. ac^2}$$

Substituting which in the third equation, and writing for $\cos. ea''$ its value obtained from the first, we have

$$e = \frac{a''}{\sin. ac} \sqrt{\left\{1 - \cos. ac^2 - \cos. a''a^2 - \cos. a''c^2 - 2 \cos. ac \cos. a''a \cos. a''c\right\}}:$$

an expression that gives the value of the least distance sought. The position of this distance is assigned by the expressions for a^{v} and d.

When the lines meet, the shortest distance between them is evidently zero; but, if a'' is not chosen at the point of intersection, this can only happen by the quantity under the radical sign becoming zero; the equation

$$0 = 1 - \cos. ac^2 - \cos. a''a^2 - \cos. a''c^2 - 2 \cos. ac \cos. a''a \cos. a''c$$

is, therefore, a condition fulfilled by every pair of lines capable of intersection. It is expressed in terms of the inclination of these lines, and of the angles which they make with a third line restricted to intersect them both: such conditions would restrict the lines to a plane, and, accordingly, the equation in question is one of the equations of condition fulfilled by lines that lie in a plane; art. 139.

PART IV.

INDETERMINATE ANALYSIS.

PRELIMINARY REFLECTIONS.

The principle that led to the ideas of the circle and the sphere admits of a more extensive application. The hypothesis may be generalized. We may assume more than one point as assigned in space, and innumerable points as connected with them by given relations. Lastly, we may investigate the nature of a surface wherein all the latter points are found.

Proceeding in this way we shall be conducted to a peculiar science, teaching to arrange lines and surfaces, not by their apparent forms, but the connection which points they contain have with other points that are given.

INQUIRIES SUGGESTED BY THESE REFLECTIONS.

Given any number of primordial elements to find the points that have assigned relations with them.

CHAPTER I.

OF LINES AND SURFACES.

SECTION I.

OF THE STRAIGHT LINE.

A straight line may be regarded as formed by an infinite number of points that have the same direction—equation of the straight line—equation of a straight line restricted to lie in a given plane—equations of lines that are parallel—when restricted to lie in a given plane—equations of lines that are perpendicular—when restricted to lie in a given plane—equation of a line that passes through a given point—equation of a line that passes through two given points—distance between a given point and line—when restricted to lie in a given plane.

213. The analysis we have used has taught us to regard the divisions of geometry as merely cases of a single problem: but whilst this view of the subject gives unity to our methods of research, it does not dispense with the necessity of discussing them under separate heads.

Even the principle of unity may be abused.

The question and its relations may be placed before our view in all their generality, but the imperfect powers of the human mind are incapable of grasping the details of an extensive problem, and require to be aided by the principles of division and arrangement.

Chap. I. Of lines and surfaces.

Art. 213. A straight line is composed of an infinite number of points that have the same direction.

Guided by these obvious laws, we have treated under many distinct divisions the great problem of the relations of space.

We have considered in detail the relations of a finite number of points, and deduced them from the principles of closed figures and primordial elements.

The relations of an infinite number of points will next engage our attention; and in studying their laws, we shall briefly employ the second of the two great instruments of inquiry to which we have alluded.

These instruments, it should be borne in mind, are not really distinct, since the principle of primordial elements is tacitly included in the theory of closed figures; and, on the other side, the theory of closed figures acts a conspicuous part in the method of research wherein primordial elements are directly employed. The occasions when either species of analysis should be adopted were explained in the second section; and as the number of points was the chief circumstance that determined this selection, it will be readily seen, on referring to the article in question, that in cases where the number of points is without limit, the method of primordial elements not only presents many advantages, but is, indeed, the only method of inquiry that can be pursued.

The problems concerning the circle and the sphere, discussed in the 4th and 5th Sections of the preceding Part, will have afforded us the first elementary notions concerning the subject, and it now only remains to generalize the analysis there used, and to examine, by the instruments employed in the preceding investigations, the properties common to all those points that have known relations to others given in space.

Sect. I. Of the straight line.

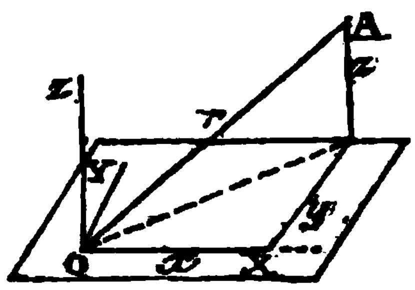
Art. 213. A straight line is composed of an infinite number of points that have the same direction.

The straight line is an assemblage of such points; and as this line is also one of the simplest examples of the kind that can be chosen, we shall commence the subject by examining anew the properties of the straight line; regarding it as an assemblage of all the points that have one common direction.

214. This last view of the subject will assist us in applying the method of primordial elements.

For since direction is merely a relation of two points, the positions of points that have the same directions can only be determined by examining them by pairs.

Fig. 232.



But assuming O, one of the points in question, as an origin, we may regard the remaining point A, as the summit of a rectangular pyramid, which is placed on the plane of the xy's, and has one of its acute angles coincident with O.

We shall then have, art. 100, for the co-ordinates of any point in the line,

$$x = r \cos. rx$$
$$y = r \cos. ry$$
$$z = r \cos. rz;$$

And as these equations are not fulfilled by the co-ordinates of any point that is not situated in the line, we may consider them, when taken conjointly, as the equations of the latter. They contain, however, a quantity r, foreign to the co-ordinates, and that must be removed before the equations shall contain merely x, y and z, as variable quantities.

Eliminating r, the equations of a straight line that passes through the origin will be,

Chap. I. Of lines and surfaces.
Art. 214. Equation of the straight line.

$$\left.\begin{aligned}\frac{x}{\cos.rx}-\frac{z}{\cos.rz}&=0\\ \frac{y}{\cos.ry}-\frac{z}{\cos.rz}&=0.\end{aligned}\right\}\quad\ldots(a)$$

Shifting the origin to a point, the distance of which from O is expressed by the co-ordinates $-\alpha$, $-\beta$ and $-\gamma$; and denoting by x', y' and z' the co-ordinates of A, as reckoned from this new origin, the formulæ of art. 98 give

$$x=x'-\alpha$$
$$y=y'-\beta$$
$$z=z'-\gamma;$$

and substituting these values in the equations a, we deduce

$$\left.\begin{aligned}\frac{x'-\alpha}{\cos.rx'}-\frac{z'-\gamma}{\cos.rz'}&=0\\ \frac{y'-\beta}{\cos.ry'}-\frac{z'-\gamma}{\cos.rz'}&=0\end{aligned}\right\}\quad\ldots\ldots b$$

which are the equations of any line in space.

Putting the equations b under the form

$$x'=\frac{\cos.rx'}{\cos.rz'}z'+\left(\frac{\alpha}{\cos.rx'}-\frac{\gamma}{\cos.rz'}\right)$$
$$y'=\frac{\cos.ry'}{\cos.rz'}z'+\left(\frac{\beta}{\cos.ry'}-\frac{\gamma}{\cos.rz'}\right)$$

and assuming

$$\frac{\cos.rx'}{\cos.rz'}=a,\ \frac{\cos.ry'}{\cos.rz'}=b,$$

$$\frac{\alpha}{\cos.rx'}-\frac{\gamma}{\cos.rz'}=a',\ \frac{\beta}{\cos.ry'}-\frac{\gamma}{\cos.\ z'}=b',$$

the equations to a straight line become

$$\left.\begin{array}{l} x' = az' + a' \\ y' = bz' + b' \end{array}\right\} \quad . \ . \ . \ . \ . \ c$$

where a, a', b and b' may have any value whatever.

To establish the truth of this last remark, we may put the equation 20, art. 100, under the form,

$$\cos. rx^2 + \cos. ry^2 + \cos. rz^2 = 1,$$

by making r coincide with r'; and this again may be written,

$$\left(\frac{\cos. rx}{\cos. rz}\right)^2 + \left(\frac{\cos. ry}{\cos. rz}\right)^2 = \tan. rz^2$$

or,

$$a^2 + b^2 = \tan. rz^2$$

where tan. rz is absolutely at our disposal: and as this equation is the only condition to which a and b are subject, we may assign those quantities at pleasure, a tangent being susceptible of any value from zero to infinity.

The arbitrary nature of a' and b' is established by a still easier process.

For, assuming z to be zero, we perceive that a' and b' are values of x' and y'; values corresponding to the intersection of the line in question with the plane of the xy's.

215. If y is zero, the line in question is contained in the plane of the xz's, and its equations are, simply,

$$\left.\begin{array}{l} x' = az' + a', \\ y = o \end{array}\right\} \quad . \ . \ . \ . \ . \ d.$$

The last of these equations is usually omitted as unnecessary, y not occurring in such cases.

When the line also passes through the origin, its equation becomes

$$x' = az',$$

where,

$$a = \frac{x'}{z'}$$

an equation that agrees with the assumption,

$$a = \frac{\cos. rx'}{\cos. rz'}$$

and teaches us that a is the cotangent of the angle formed by the lines r and x'.

From the equations of a single line we shall now pass to the problems that concern many.

216. The equations of a line being

$$\left.\begin{aligned} x &= mz + p' \\ y &= nz + q' \end{aligned}\right\} \quad \ldots\ldots e,$$

we may take as the equations of a line parallel to it,

$$x = m'z + p'$$
$$y = n'z + q',$$

where m', n', p' and q' are undetermined co-efficients, whose values must be found from the conditions of the problem.

But since it has been already shown that p and q, p' and q' are quantities depending on the position of a single point in each line, it will follow that m, n, m' and n' are, alone, sufficient to determine the directions of the latter, and, consequently, that an agreement in the directions of the lines is insured by the equations $m = m'$ and $n = n'$.

Whence, the equations of the line sought are

$$\left.\begin{aligned} x &= mz + p' \\ y &= nz + q' \end{aligned}\right\} \quad \ldots\ldots f,$$

the quantities p' and q' remaining indeterminate.

217. When the lines are restricted to lie in a plane, the sets of equations e and f become, simply,

$$x = mz + p \quad . \quad . \quad . \quad . \quad g$$
$$x = mz + p' \quad . \quad . \quad . \quad . \quad h.$$

218. The theory of parallel lines is thus made to depend upon the identity of certain co-efficients in their equations; and as these co-efficients determine the directions of the lines, it must follow that every relation of direction is capable of being reduced to an equation between the co-efficients in question. Let it be proposed, for example, to investigate the equations of a line that shall be perpendicular to a given line.

Denoting the lines by r and r', the theorem of art. 100 assures us that

$$\cos. rr' = \cos. xr \cos. xr' + \cos. yr \cos. yr' + \cos. zr \cos. zr';$$

and when the lines are perpendicular, $\cos. rr' = o$, whence

$$o = 1 + \frac{\cos. xr}{\cos. zr} \cdot \frac{\cos. xr'}{\cos. zr'} + \frac{\cos. yr}{\cos. zr} \cdot \frac{\cos. yr'}{\cos. zr'},$$

or adopting the notation used in the expressions e and f,

$$mm' + nn' + 1 = o \quad . \quad . \quad . \quad . \quad . \quad i$$

the equation of condition that exists between the co-efficients found in the equations of the lines in question.

As there are here two indeterminate quantities m' and n', and but one equation to be fulfilled, innumerable values of these co-efficients will satisfy the given condition; or, in other words, "to a given line, and from

the same point in it, innumerable perpendiculars may be drawn."

This result is, in fact, merely a preceding proposition appearing under a different aspect, since it was shown in the second part of our analysis, that, when a line is at right angles to a plane, every line drawn in the latter will be at right angles to the perpendicular.

219. Passing to the condition which limits all the lines to lie in one plane, the problem we are considering becomes more definite.

One of the axes, the axe of the y's, for example, may be taken at right angles to the given plane, and we shall then have

$$y = o, \ \frac{\cos. yr}{\cos. zr} = o, \ (xr) = \tfrac{1}{4} - (zr), (xr') = \tfrac{1}{4} - (zr')$$

$$\frac{\cos. xr}{\cos. zr'} \cdot \frac{\cos. xr'}{\cos. zr'} + 1 = o, \ \cot. zr' = -\frac{1}{\cot. zr}, m' = -\frac{1}{m}.$$

And hence, if

$$x = mz + p$$

is the equation to the given line,

$$x = -\frac{1}{m} z + p' \ldots\ldots k$$

will be the equation to any straight line that is at right angles to it.

220. Among the simple problems of the class we are considering, is that which demands the equation of a straight line restricted to pass through a given point.

In resolving this problem we reason in the following way:

Since the point is given, we know its co-ordinates, a,

β and γ. And again, to determine the line sought is to determine, in the equations

$$x = mz + p$$
$$y = mz + q,$$

the value of the co-efficients m, n, p and q.

But since the given point lies in the line we are seeking, its co-ordinates must fulfil the equations of that line, or

$$\left.\begin{aligned} \alpha &= m\gamma + p \\ \beta &= n\gamma + q \end{aligned}\right\} \ldots\ldots l.$$

Subtracting the third of these equations from the first, and the fourth from the second, we deduce for the equations of the line in question,

$$\left.\begin{aligned} x - \alpha &= m\,(z - \gamma) \\ y - \beta &= n\,(z - \gamma) \end{aligned}\right\} \ldots\ldots m;$$

and as the indeterminate quantities m and n are still found in these equations, we conclude that innumerable lines may pass through the same point.

221. If, however, a second point is given, the data will be sufficient to assign m and n, and every indeterminate quantity may, in that case, be eliminated from the equations of the line.

Assuming α', β' and γ' for the co-ordinates of the second point, we deduce, as before,

$$\alpha' = m\gamma' + p,$$
$$\beta' = n\gamma' + q:$$

and subtracting the first of these equations from the first of l, and the second from the second of l, there arises,

$$\alpha - \alpha' = m\,(\gamma - \gamma')$$
$$\beta - \beta' = n\,(\gamma - \gamma')$$

or,

$$m = \frac{\gamma - \gamma'}{\alpha - \alpha'} \quad n = \frac{\gamma - \gamma'}{\beta - \beta'},$$

substituting which, in the equations m, the latter become

$$\left.\begin{aligned} x - \alpha &= \frac{\gamma - \gamma'}{\alpha - \alpha'}(z - \gamma) \\ y - \beta &= \frac{\gamma - \gamma'}{\beta - \beta'}(z - \gamma) \end{aligned}\right\} \quad \ldots\ldots n.$$

222. The two preceding propositions enable us to determine, and express analytically, the distance between a given point and a given straight line, the last problem of the kind that it will be necessary for us to discuss.

The distance between the point and line will be measured on a second line, at right angles to the first, and passing through the given point.

But although the direct method of investigation resolves the problem by means of the two preceding it, a different mode of analysis will deduce the required results with greater facility.

The term "direct", however, must be understood in relation to the peculiar view we here take of the subject, since it would be more just to say that two methods exist, each equally direct, but proceeding on different principles. According to the first, problems are resolved by reducing them to the following elements :

1. Straight lines that pass through given points.
2. Straight lines that make given angles with other straight lines.
3. The angle between two given lines.
4. The distance between two given points.

Sect. I. Of the straight line.

Art. 222. Distance between a given point and line.

Equations or formulæ for these elements have already been obtained, and the problems can, therefore, be solved without any further reference to a geometrical analysis, by a process of the following kind.

Assuming the general equations

$$x = mz + p$$
$$y = nz + q$$

for those of the given line; and

$$x = m z' + p'$$
$$y = n z' + q'$$

for those of the line on which the required distance is measured—

Assuming, also, α, β and γ for the co-ordinates of the point, we deduce from the conditions which require the second of the lines to be at right angles to the first, and to pass through the point in question,

$$mm' + nn' + 1 = o$$
$$x - \alpha = m'(z - \gamma)$$
$$y - \beta = n'(z - \gamma)$$

whence, multiplying the second and third by m and n, adding, and having regard to the first, there arises,

$$m(x - \alpha) + n(y - \beta) + (z - \gamma) = o.$$

But the co-ordinates of the point wherein the two lines intersect will equally fulfil the equations of both; and for this point we shall, therefore, have the simultaneous equations

$$mx + ny + z = m\alpha + n\beta + \gamma$$
$$x - mz = p$$
$$y - nz = q.$$

Eliminating and putting

$$u = m(\alpha - p) + n(\beta - q) + \gamma$$
$$v = 1 + m^2 + n^2,$$

we have, for the co-ordinates of the point sought,

$$z = \frac{u}{v}$$

$$x = \frac{mu + pv}{v}$$

$$y = \frac{nu + qv}{v};$$

denoting these values by x', y' and z', and substituting them in the expression

$$r = \sqrt{\{(x' - \alpha)^2 + (y' - \beta)^2 + (z' - \gamma)^2\}}$$

we obtain, after the proper reductions, an analytical expression for the distance required.

The second method of inquiry proceeds by transforming the origin and co-ordinates, with the view of obtaining symmetrical results; and by continuing, if necessary, the transposition with an analysis by closed figures.

Thus, assuming as an easy case of the problem, α, β and γ to be zero, we have

$$u = -(mp + nq)$$

$$d = \sqrt{\{x'^2 + y'^2 + z'^2}$$

$$= \sqrt{\left\{\frac{u^2 + 2u\,(mp + nq) + (p^2 + q^2)v}{v}\right\}}$$

$$= \sqrt{\left\{\frac{(p^2 + q^2)\,v - u^2}{v}\right\}}$$

$$= \sqrt{\left\{\frac{p^2 + q^2 + (mq + np)^2}{1 + m^2 + n^2}\right\}} \quad \ldots\ldots\ o.$$

But the problem may always be reduced to this case by shifting the origin to the given point; a transformation effected by substituting for x, y and z the expressions $x - \alpha$, $y - \beta$ and $z - \gamma$. Making these substitutions, the equations

$$x = mz + p$$
$$y = nz + q$$

become

$$x = mz + p + \alpha - m\gamma$$
$$y = nz + q + \beta - n\gamma$$

and consequently the general solution will be

$$d =$$
$$\sqrt{\left\{\frac{(p+\alpha-m\gamma)^2+(q+\beta-n\gamma)^2+m\{(q+\beta-n\gamma)+n(p+\alpha-m\gamma)\}^2}{1+m^2+n^2}\right\}}$$

223. When all the points that we are considering lie in the same plane, the result is more simple; for, assuming this plane to coincide with that of the xz's, we should have $n = o$, $\beta = o$, $q = o$; and the expression for the distance would become,

$$d = \frac{p + \alpha - m\gamma}{\sqrt{1+m^2+n^2}} \;.\;.\;.\;.\;.\;.\;.\; p.$$

SECTION II.

OF THE PLANE.

Equation of the plane—equations of parallel planes—equation of a perpendicular to a plane—traces of a plane—equations of a straight line that lies in a given plane—equations of a line that is parallel to a given plane—intersection of two planes—projections of lines—the traces of a plane are at right angles to the projections of its perpendicular.

224. Straight lines and planes, inasmuch as they are the instruments used in analysing other varieties of form, necessarily present themselves at an early period in most of the arrangements that geometry permits.

The equation of the first of these two simple figures has already been deduced.

That of the second is now to occupy our attention, and may be attained by so many routes, that nearly every treatise on geometry adopts a different method of proceeding. The analysis that we shall employ is also peculiar, and deduces its conclusions from principles remarkable for their simplicity. According to this

view, we commence by transforming the system of rectangular co-ordinates, to a system in which the given plane shall coincide with one of the co-ordinate planes, that of the $z'y'$'s, for example; and we shall then have the equation

$$x' = o$$

as the equation of this plane; or, in other words, as the equation sought.

But substituting for x' its values, obtained from the formulæ for transforming the co-ordinates, art. 101, there arises

$$x \cos. xx' + y \cos. yx' + z \cos. zx' + \alpha = o \ldots q,$$

which is the equation of a plane referred to an origin, and to co-ordinates taken at pleasure.

As the line x', or the primitive axe of the x''s, is at right angles to the plane, the co-efficients, cos. xx', cos. yx' and cos. zx' are the cosines of the angles which the axes of the x's, y's and z's make with a perpendicular to the plane.

Multiplying both members of the equation q by an indeterminate factor D, it becomes,

$$x \,\mathrm{D} \cos. xx' + y \,\mathrm{D} \cos. yx' + z \,\mathrm{D} \cos. zx' + \mathrm{D}\alpha = o;$$

and, since α is arbitrary, and the cosines are merely restricted by the single condition, art. 214,

$$\cos. xx'^2 + \cos. yx'^2 + \cos. zx'^2 = 1,$$

we may assume

$$\mathrm{D} \cos. xx' = a, \ \mathrm{D} \cos. yx' = b, \ \mathrm{D} \cos. zx' = c, \ \mathrm{D}\alpha = d,$$

where a, b, c and d are altogether arbitrary. With this substitution, the equation to a plane becomes

$$ax + by + cz + d = o \ldots\ldots r.$$

The co-efficients that occur in the equation of a plane

may, also, be conveniently expressed in terms of the co-ordinates of the points wherein the plane meets the axes of the x's, y's and z's. Denoting the co-ordinates of these points by x'', y'' and z'', the form in question, which is not met with in works on this subject, may be deduced as follows.

Making y and z equal to zero, the equation q becomes

$$x'' \cos. xx' + \alpha = o;$$

whence,

$$\cos. xx' = -\frac{\alpha}{x''},$$

and making, successively, x and z, x and y, equal zero, we obtain

$$\cos. yx' = -\frac{\alpha}{y''}$$

$$\cos. zx' = \frac{\alpha}{z''}.$$

Whence, substituting in equation q, we have, for the equation of the plane,

$$\frac{x}{x''} + \frac{y}{y''} + \frac{z}{z''} = 1 \; \ldots \ldots \; s.$$

225. The relations of many planes admit of problems analogous to those which occupied our attention in the preceding section.

Thus, let it be required to determine the equation of a plane that shall be parallel to another that is given,

If the equation of the latter is

$$x \cos. a + y \cos. b + z \cos. c + \alpha = o,$$

the equation to the plane required will be

$$x \cos. a + y \cos. b + z \cos. c + \alpha' = o; \; \ldots \ldots \; t,$$

for since the same lines are perpendicular to both planes,

the angles a, b and c, art. 224, are the same in each; and as this condition is sufficient to insure the parallelism of the planes, the quantity α' may be assumed at pleasure, and is altogether independent of α.

Comparing the equations q and r, it is easy to see that when the equation of the given plane is expressed under the latter form, that of a plane parallel to it is determined by making the co-efficients bear a constant ratio to those of the given plane.

226. The next problem that will occupy our attention requires us to determine the equation of a line that shall be perpendicular to a given plane.

The plane of the $y'z'$'s being made to correspond with any plane whatever, the axe of the x''s will be a line perpendicular to the latter. The co-ordinates of a point in this axe are,

$$x = x' \cos. x'x$$
$$y = x' \cos. x'y$$
$$z = x' \cos. x'z;$$

or, eliminating x',

$$\left.\begin{array}{l} \dfrac{x}{\cos. x'x} - \dfrac{z}{\cos. x'z} = o \\[2ex] \dfrac{y}{\cos. x'y} - \dfrac{z}{\cos. x'z} = o \end{array}\right\} \quad \ldots\ldots u$$

which are the equations to the perpendicular in question.

The line we have here found passes through the origin, but it is merely necessary to decrease the ordinates by α, β and γ, in order to render the equations u applicable to any perpendicular whatever.

227. The position of a plane is often determined by that of its *traces*, or, the intersections of the plane in question with the co-ordinate planes.

To determine the "trace" on any one of the latter, the plane of the xz's, for example, we must have regard to the origin of such lines, and must reason as follows. The intersection of two planes consists of points common to them both; the equation, therefore, of either of these planes ought to be fulfilled by the co-ordinates of the points: and from this condition we learn to deduce the equations of the trace, by regarding the equations of the planes as simultaneous.

The equation of the plane of the xz's is $y = o$; and as the equation of the oblique plane is

$$x \cos. x'x + y \cos. x'y + z \cos. x'z = o,$$

we shall have, for the equations of the trace,

$$y = o$$
$$x \cos. x'x + y \cos. x'y + z \cos. x'z = o;$$

which, by mixing the equations, may be put under the more convenient form

$$y = o$$
$$x = -z \frac{\cos. x'z}{\cos. x'x} \quad \ldots\ldots v.$$

The equations of the trace on the plane of the yz's will, in like manner, be

$$x = o$$
$$y = -z \frac{\cos. x'z}{\cos. x'y} \quad \ldots\ldots x,$$

and,

$$z = o$$
$$x = -y \frac{\cos. x'y}{\cos. x'x} \quad \ldots\ldots y.$$

228. If the equations of a line are put under the form

$$a\,(x - \alpha) + c\,(z - \gamma) = o$$
$$b\,(y - \beta) + c\,(z - \gamma) = o,$$

and that of a plane under the form

$$a'\,(x - \alpha') + b'\,(y - \beta') + c'\,(z - \gamma') = o,$$

the condition which requires this line to have all its points in common with the plane, will also require that we should be able to deduce from the two first equations an equation identical with the third.

But since the line lies in the plane, we may write

$$a'\,(\alpha - \alpha') + b'\,(\beta - \beta') + c'\,(\gamma - \gamma') = o \;.\;.\;.\; z$$

which equation of condition assures us that both the line and the plane pass through a common point.

But α', β' and γ' may represent the co-ordinates of any point in the plane.

And hence, when the equation z is fulfilled, we may write, instead of the given equation of the plane, the following:

$$a'\,(x - \alpha) + b'\,(y - \beta) + c'\,(z - \gamma) = o.$$

Now multiplying the equations of the line by the indeterminate quantities m and n, adding them, and comparing the co-efficients with those belonging to the equation last deduced, we find the identity sought to require the conditions

$$am = a',\; bn = b',\; c\,(m + n) = c'\,;$$

and as these are fulfilled whenever

$$\frac{a'}{a} + \frac{b'}{b} = \frac{c'}{c} \;.\;.\;.\;.\;.\;.\; \alpha,$$

the results z and α establish the relations that exist between the co-efficients of the equations of a line, and those of the equations of the plane wherein it lies.

229. The quantities involved in α have reference only to direction; and hence that equation, considered by itself, expresses the conditions existing among the co-efficients when the line is merely parallel to the plane.

These results may be employed to determine the equation of a plane that shall be parallel to one of the axes.

Assuming, for example, the axe to be that of the y's, we shall have $x = o$, $z = o$, and y indeterminate, conditions that are deduced from the equations of the line by making a and c indeterminate, and b and α, β and γ equal to zero.

But as a and c may be varied, even when fixed values are assigned to a', b' and c', it follows that $\frac{a}{a'}$ and $\frac{c'}{c}$ change their values without any corresponding change taking place in b' and b; a result that cannot be reconciled with the equation α, unless the fraction $\frac{b'}{b}$ is capable of an infinity of values, or, in other words, unless it has the indeterminate form $\frac{0}{0}$.

The co-efficient b' is thus shown to have the value zero, and the equation of the plane becomes,

$$a'(x - \alpha') + c'(z - \gamma') = o \;.\;.\;.\;.\;.\; \beta$$

a result that proves itself, since the equation of the plane must evidently be independent of the co-ordinate to which the plane itself is parallel.

230. The intersection of two planes consisting of points that lie in both, will be obtained by uniting their equations.

This proposition gives occasion to a remark that will

be useful in the demonstration of the proposition next following; it may be thus expressed: The equation of any plane parallel to the zx's is $y =$ constant; and uniting this equation with the last deduced, the latter still remains

$$a'(x - \alpha') + c'(z - \gamma') = o,$$

whence we may regard this result as belonging to the intersection of two planes, namely, of a plane parallel to the xy's and a plane parallel to the axe of the y's.

231. The orthographic projection, or, as it is here more simply termed, "the projection of a line on a plane," is, art. 95, the intersection of the latter with a second plane, at right angles to the first, and passing through the line.

Assuming the equations of the line to be the same as in art. 228, the equations of the projections on the three co-ordinate planes will, by the preceding article, be respectively,

$$\begin{aligned} a'(x - \alpha') + c'(z - \gamma') &= o \\ b'(y - \beta') + c'(z - \gamma') &= o \\ a'(x - \alpha') + b'(y - \beta') &= o. \end{aligned}$$

But from the remark in art. 228, α, β and γ may be substituted for α', β' and γ'; and, moreover, in establishing the identity of the equations of the line with the plane represented by the first of the above equations, art. 228, we shall have

$$n = o;$$

whence, the equation α will be satisfied, if $a = a'$, and $c = c'$: and in a similar manner we discover that $b = b'$, from which, the three preceding equations appear identical with those of the line itself; and we conclude, that

when the equations of a line are so arranged that in each an ordinate is wanting, any one of the equations represents a projection of the line on one of the co-ordinate planes; on the plane, namely, which is at right angles to the ordinate wanting.

232. The equations u of a perpendicular to a plane are,

$$\frac{x}{\cos. x'x} - \frac{z}{\cos. x'z} = 0$$

$$\frac{y}{\cos. x'y} - \frac{z}{\cos. x'z} = 0$$

the first of which belongs to the projection on the plane of the xz's and the second to the projection on the plane of the yz's: comparing these with the equations of the traces, art. 227, and with the equations of lines that are perpendicular, art. 219, we discover "the traces of a plane to be at right angles to the projections of its perpendicular."

PRELIMINARY REFLECTIONS TO SECTIONS III. AND IV.

In investigating the relations of a definite number of points, the number is, itself, a character, in terms of which the analysis can be arranged. But as this principle of classification manifestly fails when the number of points is infinite, we have yet to supply that deficiency.

Now the analysis of an infinite number of points proceeding by the equations they give rise to, we may adopt a principle of classification extensively used in algebra, and arrange the several steps of the process by the degrees of the resulting equations.

INQUIRY SUGGESTED BY THESE REFLECTIONS.

To discover the curve, or surface, formed by all those points the relations of which to known elements shall be expressed in an equation of the second degree.

SECTION III.

OF PLANE LINES OF THE SECOND ORDER.

Points which lie in a plane are assigned by reference to two primordial elements—of the parabola—of the ellipse—of the hyperbola—connection of the equations discussed in this section—the curves discussed in this section referred to oblique co-ordinates—conjugate diameters—equation of the hyperbola referred to its asymptotes—polar equations of the curves discussed in this section—every equation of the second degree between two variables will be fulfilled by the co-ordinates of one or other of the curves discussed in this section.

233. Lines, we have seen, are formed of an infinity of points, whose relations to known primordial elements are restricted by invariable conditions.

The circle, a line examined in detail in the Fourth Section of Part III., is the locus of all points that, lying in a plane, have a constant distance from a given point in space.

The most general equation of a circle, art. 99, is

$$(x-\alpha)^2 + (y-\beta)^2 = \gamma^2 \; \ldots\ldots \; a;$$

but arranging the primordial elements in such a manner

that the origin shall fall at the given point, this equation reduces to

$$x^2 + y^2 = \gamma^2 \; \ldots\ldots \beta$$

whence all the properties have already been developed.

But the relation which has led to this simple and important curve is only a particular case of the problem enunciated at the commencement of the present Chapter.

It is there proposed to express the positions of the points that have assigned relations to any number of primordial elements.

But as the inquiry taken in this wide sense extends beyond the limits of a section, further subdivisions become neeessary; and the Preliminary Remarks affixed to the Chapter we are now engaged upon, teach us to seek them in the relations existing among the points assigned in space, and those whose position is the object of our analysis.

The degree of the equation whereby these *relations* are expressed, although not an unexceptionable principle of classification, is one that custom has sanctioned, and which possesses many advantages.

The case when this equation is of the first degree has already occupied our attention, and we shall, therefore, proceed in the present section to investigate the positions of points whose connection with the assigned primordial elements can be expressed by a quadratic equation.

These points, it will be observed from the heading of the section, are regarded as lying in one plane, and, consequently, their position will be completely assigned by referring them to two primordial elements.

If the co-ordinates that constitute this reference are given, either explicitly or implicitly, the question re-

gards a single point in space, whose position is then assigned.

But if in place of the co-ordinates themselves we have a relation between them, the points are only partially assigned, and may be infinite in number.

234. Thus assuming as primordial elements a line and a point; and as co-ordinates the distances from these elements; we may propose to investigate the positions of all the points for which the two co-ordinates are equal.

Let us denote the line by m, the indefinite point by P, and the point that serves as a primordial element by the letter A.

The distance from A to P is, art. 99,

$$\sqrt{(x-\alpha)^2+(y-\beta)^2}$$

and the distance from P to the line is, art. 223,

$$\frac{p+x-my}{\sqrt{1+m^2+n^2}};$$

whence we have for the equation to the locus of the point sought,

$$\sqrt{\{(x-\alpha)^2+(y-\beta)^2\}}=\frac{p+x-my}{\sqrt{1+m^2+n^2}}\ \ldots\ldots\gamma$$

But the principle of symmetry will guide us to an equation of a simpler form, and teach us to choose the primordial elements in a manner that shall cause all the given parts of the same kind to enter alike.

Thus, since no reason can be assigned why the axe of the x's should pass "above" rather than "below" the point A, let us draw it through that point: and since there is no reason why this axe should make with m an acute angle on one side rather than on the other, let us

draw the axe so as to make the angles equal on either side, that is, so as to be perpendicular to m. Again, since the nature of the proposition does not require the position of the origin to be influenced to a greater extent by one rather than by the other of the given elements, let it be assumed intermediate between them.

With these conventions we have,

the distance of P from $A = \sqrt{\{(x-a)^2+y^2\}}$
the distance of P from $M = x + a$,

which give for the equation of the curve

$$-x+a=\sqrt{\{(x-a)^2+y^2\}};$$

or, squaring and cancelling like quantities,

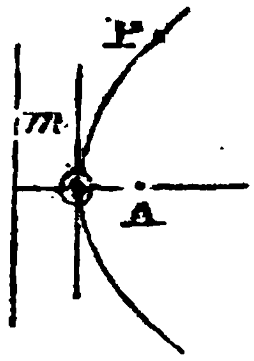
$$y^2 = 4\,ax. \;.\;.\;.\;.\;\delta$$

The curve expressed by this equation is called a *parabola*, and its properties may be investigated, as in the cases of the straight line and the circle, from the equation itself.

The analysis will stand as follows:

Making $x = o$, we have $y = o$; whence, the curve passes through the origin. And since for any positive value of x, from zero to infinity, the value of y is positive; and since, also, the value of y increases with that of x, the curve, we conclude, has, on the positive side, two equal and infinite branches that continually recede from the axe of the x's.

Fig. 233.



When x is negative, y is imaginary; no part, therefore, of the curve lies on the negative side of the origin.

These results are alone sufficient to afford some idea of the curve investigated; but it will not be difficult to obtain others that assist in defining its character.

The equation

$$y = ax + b,$$

to pursue this subject, being the general equation of a straight line, may, by due values assigned to the co-efficients, be made to represent a line that has any assigned position with respect to the curve; and, consequently, to represent a line that shall pass through a given point in the parabola and agree as nearly with the curve at that point, as a straight line can do.

But on comparing the equation of this line with the equation of the parabola, we observe the increase of y to follow different laws: the y of the straight line is found, for equal increments of x, to increase more rapidly than the y of the curve. And we conclude, that, as x increases, the parabola bends away from the line in question and approaches the axe of the x's; or, in other words, the curve, in every part of its course, continues concave to that axe.

Assuming $x = \alpha$, we have

$$y^2 = 4\,\alpha^2$$

or,

$$2y = 4\,\alpha.$$

Representing this double ordinate by the letter p, the equation to the parabola becomes

$$y^2 = px. \quad . \ . \ . \ . \ . \ . \ . \ \varepsilon$$

The great attention formerly bestowed on this and the remainder of the curves described in the present Section, has caused every line drawn in or about them, and all the elements to which they refer, to receive some peculiar name. More extended views of geometry have, in a great measure, rendered this phraseology obsolete,

A choice of the primordial elements, proceeding on the principle used in the last problem would lead, however, to an expression that admits of a more ready development.

According to that principle, the situation of the primordial elements must be symmetrical with respect to the elements that serve as data.

To obtain this symmetry, one of the axes, the axe of the x's, for example, should pass through the two given points; the remaining axe should be intermediate between the latter, and have, at the same time, a direction at right angles to the axe first mentioned.

With these conditions, $\alpha = -\alpha'$, and β and β' are zero; whence the equation of the curve sought becomes,

$$\sqrt{\{(x-\alpha)^2+y^2\}} \pm \sqrt{\{(x+\alpha)^2+y^2\}} = 2a\,;$$

and by carrying the first of the radical expressions to the right hand member, squaring and reducing, there results,

$$y^2 = \frac{a^2-\alpha^2}{a^2}(a^2-x^2) \quad \ldots\ldots \eta$$

an equation that belongs to either of the two problems whose solutions we have embraced in this analysis.

Recollecting, however, that in the first problem $2a$ is the sum of two sides of a triangle whose third side is 2α; and that in the second $2a$ is the difference of those sides; we shall readily perceive that $a^2-\alpha^2$ is positive in the former case, and negative in the latter. And the equation above deduced, to be free from imaginary expressions, must, in the two cases alluded to, assume distinct forms, being written,

$$y^2 = \frac{a^2-\alpha^2}{a^2}(a^2-x^2)$$

when a represents the sum of the sides, and

These branches, we are further assured, altering their course with as much regularity as the corresponding branches of a circle, present no sinuosities; for, were it otherwise, the value of y would decrease irregularly with the increase of x; a supposition inconsistent with the equation we have deduced.

The curve here investigated is called the *ellipse*; the points A and B are termed the *foci*; and the lines HI and KL the *major* and the *minor axis*.

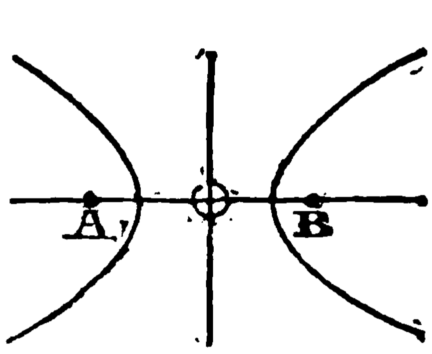
236. The equation

$$y = \frac{b}{a}\sqrt{(x^2 - a^2)},$$

when analysed by a similar process, will lead to results somewhat different, and present us with a curve that has been termed the *hyperbola*.

The imaginary values of y correspond, in this case, to values of x less than a, and are not found when x has values between a and infinity. The quantities x and y, in this, as in the preceding case, enter the equation by their second powers only, and thus indicate the existence of equal branches on either side of the origin. Combining these results, we perceive the hyperbola to consist of four equal and infinite branches, that are turned in opposite directions, and continually recede from the axe of x's.

Fig. 235.



When x is very great, the equation

$$y^2 = \frac{b^2}{a^2}(x^2 - a^2)$$

approaches to the equation

$$y^2 = \frac{b^2}{a^2}\ x^2 \ldots\ldots$$

served, is equal to $2a$, and the minor to $2b$; for since I is a point in the curve, AI — IB is equal to $2a$; and as the curves on either side of the centre are symmetrical, IB and AH are equal, and AI — IB is the same as HI; or, in other words, the major axis HI is equal to $2a$. Again, substituting in the equation

$$y - \frac{b}{a}x = o,$$

OI or a, for x, we have $y = \mathrm{I}r$: but the value of y resulting from the substitution in question is b, whence, the minor axis $r\mathrm{I}$ is equal to $2b$.

237. The object of the present Section, we recollect, was to investigate curves capable of being represented by equations of the second degree. The four curves that we have analysed, the circle, the parabola, the ellipse and the hyperbola, possess this property; but it becomes a question whether the equations we have used to represent the curves discussed in this Section, might not be deduced from each other; and whether, under all the forms they admit, these equations still represent the same curves.

The equation

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

will be very proper for this inquiry, since we have seen, that, according as b^2 is positive or negative, it represents either an ellipse or an hyperbola.

Now, substituting for x, $x - a$, or, what amounts to the same, shifting the origin by the distance a; the equation becomes

and as y cannot, in this case, be real, unless x is zero, the curve becomes a point, whose co-ordinates are

$$y = 0, x = 0.$$

This last result proceeds upon the supposition of λ^2 being positive; but if, on the contrary, it has a negative value, the equation assumes the form

$$y^2 = \lambda^2 x^2,$$

or,

$$(y - \lambda x)(y + \lambda x) = 0,$$

or,

$$y - \lambda x = 0$$
$$y + \lambda x = 0,$$

and indicates, as we have before observed, two straight lines.

If b^2 and a^2 are both negative, the equation may be written

$$y^2 = -\frac{b^2}{a^2}(a^2 + x^2)$$

and as for every real value of x, y is here imaginary, the equation has no geometrical representation; or, to use the language of the analyst, the equation belongs to an imaginary curve.

Thus, examined under every form which it is capable of assuming, the equation

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

is found to present, in a geometrical point of view, the following cases:

It designates, 1. A closed curve, which is either a circle or an ellipse.

2. A curve, the hyperbola, composed of two pair of

opposite branches, each pair enclosing a space on all but one side.

3. A curve, the parabola, intermediate in its properties between the ellipse and hyperbola; that encloses a space on all sides but one, but that does not possess two pair of branches.

4. A straight line, or a pair of straight lines.

5. A point.

6. An imaginary curve.

238. We* will now seek the systems of oblique co-ordinates, relative to which the curves we have investigated preserve an equation of the same form as when analysed in relation to their axes. For this purpose, we will employ the formulæ of transformation

$$x = x' \cos. xx' + y' \cos. y'x,$$
$$y = x' \sin. xx' + y' \sin. y'x;$$

By substituting these values of x and y in the equation

$$a^2y^2 + b^2x^2 = a^2b^2,$$

or,

$$\left(\frac{y}{b}\right)^2 + \left(\frac{x}{a}\right)^2 = 1,$$

it becomes,

$$(a^2 \sin. y'x + b^2 \cos.^2 y'x)\, y'^2 + (a^2 \sin.^2 x'x + b^2 \cos.^2 x'x)\, x'^2 + 2\,(a^2 \sin. x'x \sin. y'x + b^2 \cos. x'x \cos. y'x)\, x'y' - a^2b^2 = 0.$$

But for this equation to be of the same form as the equation relative to the axes, it is necessary that it should not contain the term which is multiplied into $x'y$; or, in other words, we must so choose the angles $(x'x)$ and $(y'x)$, that, when substituted in the preceding equation,

* Part of the three following articles is from Biot's Analytical Geometry.

these values shall cause the co-efficient of $x'y'$ to become zero; which gives the equation

$$a^2 \sin. xx' \sin. y'x + b^2 \cos. x'x \cos. y'x = 0.$$

This condition between the angles $(x'x)$ and $(y'x)$, is not sufficient to determine the latter, but merely suffices to render one of them known when the other is given; and as one of the angles may thus be chosen at pleasure, it follows that an infinity of oblique axes may be found that have the property in question; and since the equation we have deduced gives

$$\tan. x'x = -\frac{b^2}{a^2}\cot. y'x,$$

the systems will be real.

Making successively $y' = o$, and $x' = o$, we have the distances from the origin of the co-ordinates to the points in which the curve cuts the axes.

And if we represent these distances by a' and b', the first being reckoned on the axe of the x's, and the second on that of the y's, we find

$$a'^2 = \frac{a^2b^2}{a^2 \sin.^2 xx' + b^2 \cos.^2 x'x}$$

$$b'^2 = \frac{a^2b^2}{a^2 \sin.^2 y'x + b^2 \cos.^2 y'x}$$

and the equation of the curve becomes,

$$a'^2y'^2 + b'^2x'^2 = a'^2b'^2$$

or,

$$\left(\frac{y'}{b'}\right)^2 + \left(\frac{x'}{a'}\right)^2 = 1 \; \ldots\ldots \; x.$$

239. The form of this equation being precisely that

belonging to the curve referred to rectangular axes, it follows here, also, that y will have the same values when measured on either side of the x's, and conversely. Now any line that bisects, in this way, the ordinates of a curve, is called a *diameter;* and when the property is mutually possessed, as in the present case, by the axes of the x's and the y's, these lines are termed *conjugate* diameters.

The lines $2a'$ and $2b'$ are the conjugate diameters, to which the curve is referred when the positions of its points are expressed by the equation x.

If we multiply together the values of a'^2 and b'^2, and have regard to the equation

$$a^2 \sin.\, y'x \sin.\, x'x + b^2 \cos.\, y'x \cos.\, x'x = 0,$$

there will result,

$$a'b' = \frac{ab}{\sin.(y'x - x'x)},$$

where the angle $(y'x - x'x)$ is that formed by the conjugate diameters.

This value of $a\,b$ being substituted in the expressions for a'^2 and for b'^2, the latter will give,

$$b'^2 \sin.^2 (y'x - x'x) = a^2 \sin.^2 x'x + b^2 \cos.^2 x'x$$
$$a'^2 \sin.^2 (y'x - x'x) = a^2 \sin.^2 y'x + b^2 \cos.^2 y'x.$$

These equations may be put under the form

$$b'^2 \sin.^2 (y'x - x'x) = a^2 \sin.^2 x'x (\sin.^2 y'x + \cos.^2 y'x) + b^2 \cos.^2 x'x (\sin.^2 y'x + \cos.^2 y'x)$$

$$a'^2 \sin.^2 (y'x - x'x) = a^2 \sin.^2 y'x (\sin.^2 x'x + \cos.^2 x'x) + b^2 \cos.^2 y'x (\sin.^2 x'x + \cos.^2 x'x),$$

whence, adding and effecting the multiplication indicated, we have,

$(a'^2 + b'^2) \sin.^2(y'x - x'x) = (a^2 + b^2)(\sin.^2 x'x \cos.^2 y'x + \cos.^2 x'x \sin.^2 y'x) + 2a^2 \sin.^2 x'x \sin.^2 y'x + 2b^2 \cos.^2 x'x \cos.^2 y'x,$

where the co-efficient of $a^2 + b^2$ wants but a term in order to be equal to $\sin.^2(y'x - x'x)$: adding this term, the equation becomes

$$(a'^2 + b'^2) \sin.^2(y'x - x'x) = (a^2 + b^2) \sin.^2(y'x - x'x)$$
$$+ 2\sin. x'x \sin. y'x\,(a^2 \sin. x'x \sin. y'x + b^2 \cos. x'x \cos. y'x)$$
$$+ 2\cos. x'x \cos. y'x\,(a^2 \sin. xx' \sin. y'x + b^2 \cos. x'x \cos. y'x)$$

The part which is independent of $\sin.(y'x - x'x)$ vanishes of itself, in consequence of the condition existing between $(y'x)$ and $(x'x)$; and hence,

$$a'^2 + b'^2 = a^2 + b^2.$$

The three equations

$$\left.\begin{array}{l} a^2 \tan. x'x \tan. y'x + b^2 = 0 \\ ab = a'b' \sin.(y'x - x'x) \\ (a^2 + b^2) = (a'^2 + b'^2) \end{array}\right\} \ldots\ldots \lambda$$

suffice to determine the conjugate diameters when the axes are known.

240. If, in place of destroying the term which contains the product of x' and y' we seek, in the transformed equation, page 474, to eliminate the terms that contain the squares of those quantities, we shall have the equations

$$a^2 \sin.^2 y'x + b^2 \cos.^2 y'x = 0$$
$$a^2 \sin.^2 x'x + b^2 \cos.^2 xx' = 0,$$

which give

$$\tan. y'x = \pm\sqrt{-\frac{b^2}{a^2}}$$

$$\tan. x'x = \pm\sqrt{-\frac{b^2}{a^2}};$$

Taking the plus sign in the first of these results, and the minus sign in the second, and assuming

$$b = b\sqrt{-1},$$

the curve becomes the hyperbola, art. 236, and the new axes are its "asymptotes," or the lines, art. 236, to which it continually approaches. But with these values of tan. $y'x$, tan. $x'x$ and b, the transformed equation reduces to

$$x'y' = \frac{a^2 + b^2}{4},$$

which is the equation of the hyperbola referred to its asymptotes.

241. The equation of the ellipse

$$a^2y^2 + b^2x^2 = a^2b^2$$

has served as a type to which we have referred the equations of each of the curves treated of in this Section.

The ellipse, the reader will recollect, is formed of all the points possessing a certain property; the distance of the points is estimated from two given foci, and for each point the sum of these distances is a constant quantity.

Now, calling one of these distances y' we might seek the relation between y' and the angle which it makes with a known line, the major axis, for example.

Such a relation, art. 91, would form a polar equation to the ellipse, the pole being, in this case, the focus.

The polar equation will, therefore, be obtained by substituting in the expression

$$a^2y^2 + b^2x^2 = a^2b^2$$

the values

$$y = y' \sin. y'x,$$
$$x = y' \cos. y'x - a,$$

or, we may obtain the polar equation of the ellipse by

recurring to the proposition from which our knowledge of the curve is derived. The two given points A and B, and the point which is sought, P, are the elements of position whose relations constitute the object of research. Now putting 2α for the distance between A and B; y' for the distance of P and A; y'' for that of P and B; and, finally, $2a$ for the sum of y' and y'', we have from the equations deduced for the relations of three points,

$$\begin{aligned} y'' \sin. y''\alpha - y' \sin. y'\alpha &= 0 \\ y'' \cos. y''\alpha + y' \cos. y'\alpha &= 2\alpha \\ y'' + y' &= 2a; \end{aligned}$$

and equating the values of $\sin.^2 y''\alpha$, obtained from the first and second of these equations, and substituting for y'' its value obtained from the third, we deduce

$$y' = \frac{a^2 - \alpha^2}{a - \alpha \cos. y'\alpha},$$

or, putting

$$\frac{\alpha}{a} = e, \ (y'\alpha) = \theta,$$

$$y' = a \frac{1 - e^2}{1 - e \cos. \theta},$$

which is the polar equation sought.

The angle θ is usually taken as the supplement of the angle here designated by that letter, and the equation is, therefore, more frequently written

$$y' = a \frac{1 - e^2}{1 + e \cos. \theta} \ \ldots\ldots \ \mu.$$

As the quantity $a^2 - \alpha^2$ is the same that we have denoted, art. 235, by b^2, when the curve changes into the hyperbola, this quantity becomes negative, and the polar equation of the hyperbola will be

$$y' = a \frac{e^2 - 1}{1 + e \cos. \theta} \ \ldots\ldots \ \nu,$$

and, in like manner, since the ellipse changes into a parabola, art. 237, when $\frac{a^2 - \alpha^2}{a}$ becomes 2β, and $\frac{\alpha}{a}$ or e becomes unity, we deduce for the polar equation of the parabola,

$$y' = \frac{2\beta}{1 + \cos. \theta} \quad \ldots\ldots\ldots \xi.$$

242. These results might, with propriety, terminate the analysis of the curves that have formed the subject of the present Section; but as the principle of arrangement on which it proceeds classes together curves whose equations are of the second degree, it will be proper, before we dismiss this subject, to examine whether *all* the curves that can be so expressed, are included under one or other of the preceding forms.

242—2. This inquiry will be facilitated by previously adapting the formulæ of art. 101, to the case wherein the primitive co-ordinates are oblique, and the new co-ordinates rectangular.

Denoting the former system by x and y and the latter by x' and y', we have, art. 101,

$$x' = x \cos. xx' + y \cos. yx',$$
$$y' = x \sin. xx' + y \sin. yx';$$

and, eliminating y,

$$x = \frac{x' \sin. yx' - y' \cos. yx'}{\sin. yx};$$

and, eliminating x,

$$y = \frac{y' \cos. xx' - x' \sin. xx'}{\sin. yx}.$$

When the axes of x and x' coincide, these equations become $x = x' - y' \cot. yx$, $y = y' \operatorname{cosec}. yx$.

Sect. III. Of plane lines of the second order.

Art. 242. Every equation of the second degree between two variables will be fulfilled by the co-ordinates of one or other of the curves discussed in this Section.

242—3. As the formulæ here obtained will enable us to extend the results deduced for the particular case where the primitive co-ordinates are rectangular, to the general problem, in which the co-ordinates are inclined at any angle, it will be a convenient method of analysis to commence with the former system of primordial elements.

Assuming, therefore,

$$Py^2 + Qx^2 + Rxy + Sy + Tx + U = 0$$

as a general quadratic, expressing the relation between two rectangular co-ordinates, we must employ the formulæ

$$\begin{aligned} y &= x' \sin. xx' + y' \cos. xx' + \beta \\ x &= x' \cos. xx' - y' \sin. xx' + \alpha \end{aligned}$$

to get rid of the terms which do not enter in the form required.

The object, it will be recollected, is to reduce the general quadratic to the form

$$a^2y^2 + b^2x^2 = a^2b^2;$$

and as terms enter involving the first powers of x and y, and the product of their first powers, it would be necessary, were it intended to eliminate those terms at one operation, to retain three of the arbitrary constants belonging to the equations of transformation.

But as two eliminations are, in this case, more commodious than one, we shall merely retain such constants as suffice to remove the terms in x and y.

To effect this, assume

$$\begin{aligned} y &= (y' + \beta) \\ x &= (x' + \alpha); \end{aligned}$$

the terms in y' will be found, by multiplying the terms

Multiplying by R, and substituting for R^2 its value 4PQ, this equation reduces to

$$RT - 2QS = 0;$$

and hence we conclude, that, when α and β assume the form $\frac{0}{0}$, the equation is that of a line.

The general quadratic, therefore, either belongs to a parabola or a line, or it can be reduced to the form

$$P'y^2 + Q'x^2 = -U';$$

and it only remains for us to discuss the cases which this transformed equation admits.

Multiplying each side by an indeterminate factor X, and comparing the result with the equation

$$A^2y^2 + B^2x^2 = A^2B^2$$

the comparison gives

$$P'X = A^2,\ Q'X = B^2 = -U'X = A^2B^2,$$

or,

$$A^2 = -\frac{U'}{Q'}$$

$$B^2 = -\frac{U'}{Q'} \cdot \frac{Q'}{P'},$$

which, denoting the numerator of U′, page 482, by — 4NP, may be written,

$$A^2 = \frac{N}{4P'Q'^2}$$

$$B^2 = \frac{N}{4P'Q'^2} \cdot \frac{Q'}{P'}.$$

Hence, if N is positive, A^2 will be positve; and B^2

effect, is to introduce an arbitrary constant, when the equation would belong, not to the axe of the x's alone, but to any line parallel to that axe.

It results from this investigation that every quadratic between rectangular co-ordinates, belongs to one or other of the curves described in the first part of this Section; but it remains yet to be established that such is also the case when the co-ordinates are oblique.

The demonstration, however, will immediately follow from what has been said; for since the formulæ of 242—2 contain merely sin. xy as their denominator, the given equation, when transformed to an equation between rectangular co-ordinates, cannot, since the denominator does not admit the value zero, have ambiguous co-efficients; and the reasoning used in the preceding investigation may, therefore, be extended, without fear of error, to the equation so transformed.

The conditions that determine the class of curves to which the equation belongs will, also, be the same in the two cases; but this remark must not be extended to the subdivisions of those classes.

We may assume, to demonstrate this truth, the general equation as existing between oblique, and the transformed equation between rectangular co-ordinates; distinguishing the co-ordinates of the latter by accents, we shall then have, art. 242—3,

$$P' = P \text{ cosec. } yx^2 - R \text{ cosec. } yx \text{ cot. } yx + Q \text{ cot. } yx^2$$

$$Q' = Q,$$

$$R' = R \text{ cosec. } yx - 2\,Q \text{ cot. } yx\,;$$

whence,

SECTION IV.

OF SURFACES OF THE SECOND ORDER.

Surfaces arranged by the degrees of their equations—equation of the cylinder—sections of the cylinder—equation of the cone—sections of the cone—surfaces of the second degree—ellipsoid—hyperboloid of one sheet—hyperboloid of two sheets—paraboloid.

243. The problems that presented themselves, when treating of the points that lie in a plane we sought their simplest combinations, are connected with analogous problems relative to points in space.

The sphere, for example, holds among the latter class of problems the place assigned in plane geometry to the circle. And, by pursuing the course of investigation which is thus suggested, we might readily extend the propositions of the preceding Section, and render them applicable to points not restricted to lie in a plane.

But the principle of arrangement adopted in the Section alluded to, had reference, not to the number of elements, but to the degree of the resulting equation; and this principle we shall still follow; leaving ourselves, in

being everywhere circles of the same radius, the symmetry which is observed in the circle will extend to the cylinder; and this surface, infinitely extended toward either extremity of the axis, will be every where at the same distance from that line.

The form of an oblique section will thus depend upon the inclination of the trenchant plane to the axis of the cylinder, but will be independent of the angle it forms with lines at right angles to the axis.

Now, assuming for y the value

$$y = y' \cos. yy',$$

we cause the plane of the $x'y'$'s to assume any required position with regard to the axe of the z's, and, consequently, to meet the surface in a line which may be made, by a proper assumption of (yy'), to correspond with any intersection whatever. The form of an oblique section will, therefore, be investigated in the most general way by determining the points wherein the plane of the $x'y'$'s meets the surface.

As such points are common both to the cylinder and the plane, their positions will be found by combining the equations of these surfaces.

But the equation of the plane is $z' = o$, and as z' does not enter the equation of the surface, this last equation must equally belong to the surface and to the intersection whose form is the object of research.

The equation of the section will, therefore, be

$$x'^2 + y'^2 \cos.^2 yy' = r^2;$$

which belongs to an ellipse.

246. Let us now analyse the equation of the surface that contains all the straight lines which pass through

By assuming

$$z = z' + h, \qquad x = x', \qquad y = y',$$

the origin will be transported to a point in the axe of the z's, distant by h from the point that was given. But the equation then becomes

$$x'^2 + y'^2 = (z' + h)^2 \tan.^2 zm;$$

and making $z = o$, with the view of obtaining the intersection of the surface with the plane of the $x'y'$'s, we obtain for this line

$$x'^2 + y'^2 = h^2 \tan.^2 zm \quad . \quad . \quad . \quad . \quad .$$

which indicates a circle, the radius of which is h tan. zm.

The result here given will be obtained whether h is positive or negative; and as the radius of the circle increases in the same proportion as z, we conclude the surface to consist of two infinite branches, or *curved sheets*, which only partially enclose a space, presenting an opening towards either extremity of the given line.

The solid enclosed by this surface is the common cone, art. 3 and 9; the given point is the centre, which has been improperly termed the apex; and the given line is an axis, around which either sheet of the surface is symmetrically disposed.

247. To determine the oblique section of a cone with a plane, is a problem that may be resolved according to the method used in determining the sections of the sphere.

For, by transforming the co-ordinates, in the preceding equations of the cone, we render their position arbitrary, and, consequently, such as will enable us to make either of the co-ordinate planes coincide with any plane that can be given.

which transforms the equation ϖ, of the cone, into the following,

$$y'^2 + x'^2 \{\cos.^2 xx' - \sin.^2 xx' \tan.^2 zm\} + 2x' \{\alpha \cos. xx' - \gamma \sin. xx' \tan.^2 zm\} + \alpha^2 - \gamma^2 \tan.^2 zm = o,$$

an equation that can be made to coincide with

$$a^2y^2 + b^2x^2 = a^2b^2,$$

by getting rid of the term which contains the first power of x'.

Equating the co-efficient of this term with zero, there results,

$$\gamma^2 \tan.^2 zm = \alpha^2 \cot.^2 xx' \cot.^2 zm \quad \ldots\ldots \sigma$$

and making this substitution in the known term of the equation, and observing that

$$\cos.^2 xx' - \sin.^2 xx' \tan.^2 zm = 1 - \sin.^2 xx' \sec.^2 zm$$

the equation in question reduces to

$$y'^2 + x'^2 (1 - \sin.^2 xx' \sec.^2 zm) + \alpha^2 (1 - \cot.^2 xx' \cot.^2 zm) = o.$$

Now this equation can only become of the form

$$a^2y'^2 + b^2x'^2 = a^2b^2,$$

when, either it is already under that form, or has departed from it by the loss of a common factor. Let us suppose this factor X; and, multiplying by it, equate the resulting co-efficients with those of the equation which it is intended to reduce.

We should then have

$$a^2\text{X} = 1$$

$$b^2\text{X} = 1 - \sin.^2 xx' \sec.^2 zm$$

$$a^2b^2\text{X} = \alpha^2 (\cot.^2 xx' \cot.^2 zm - 1);$$

whence, multiplying the two first expressions, and comparing the product with the third, there results,

$$\text{X} = \frac{1 - \sin.^2 xx' \sec.^2 zm}{\alpha^2 \cos.^2 xx' \cot.^2 zm - 1};$$

To avoid this difficulty, we may substitute for these ambiguous expressions others, whose values will be more definite. The equation σ gives

$$\gamma^2 \tan.\ ^2zm = \frac{\cot.\ ^2zm}{\cot.\ ^2x'z}\ \alpha^2;$$

and for the particular case that we are considering, which assumes $x'z = \frac{1}{4}$, we have, from the equations ρ, $z = \gamma$; whence, γ must be finite for finite values of z; and as $\frac{\cot.\ zm}{\cot.\ x'z}$ is not finite, the identity that exists between the sides of the preceding equation will require that α should be equal to zero.

Attending to these remarks, the values of a and of b may each be reduced to

$$\gamma \tan.\ zm,$$

and the section is a circle, having this quantity for the value of its radius.

To complete the investigation, let us examine the conclusions that follow from assuming $\alpha = o$, and $\frac{\cot.\ zm}{\cot.\ x'z}$ finite. It is, in the first place, evident, from the equation

$$\gamma \tan.\ zm = \frac{\cot.\ zm}{\cot.\ zx'}\ \alpha,$$

that as γ is then equal to zero, the trenchant plane will pass through the given origin; or, in other words, through the given point: a further examination separates this case into two subdivisions, according as $(x'z)$ is greater or less than (zm).

On the first hypothesis, the equation reduces to

$$y'^2 = (1 - \sin.\ ^2xx'\ \sec.\ ^2zm) \times (-x'^2);$$

and as y will here be imaginary for every value of x

except $x = o$, the section will be a point, the co-ordinates of which are $y = o$ and $x = o$; in other words, it will be the origin itself.

Upon the second hypothesis, the equation assumes the form

$$y'^2 - \{1 - \sin.^2 xx' \sec.^2 zm\} x'^2 = o,$$

or,

$$\{y' - x' \surd (1 - \sin.^2 xx' \sec.^2 zm)\}$$
$$\{y' + x' \surd (1 - \sin.^2 xx' \sec.^2 zm)\} = o,$$

and as this equation may be satisfied by equating either of the factors, separately, with zero, we have

$$y' - x' \surd \{1 - \sin.^2 xx' \sec.^2 zm - 1\} = o,$$
$$y' + x' \surd \{1 - \sin.^2 xx' \sec.^2 zm - 1\} = o,$$

which indicate for the section two straight lines that pass through the origin, and form with the axis of the cone, the angles

$$\cos.^{-1} \left\{ \frac{\cos.\ zm}{\cos.\ zx'} \right\} \qquad -\cos.^{-1} \left\{ \frac{\cos.\ zm}{\cos.\ xz'} \right\}.$$

Had we assumed $(zx') = (zm)$, the co-efficient of x'^2, in the general equation of the cone, would have become zero; and, as α and γ would still have been indeterminate, we might have chosen them in such a manner as to destroy the final term, or, to make

$$\alpha^2 = \gamma^2 \tan.^2 zm;$$

from this equation we should have derived

$$\alpha = \pm \gamma \tan.\ zm.$$

Substituting the second of these values in the general equation alluded to, it would become,

$$y'^2 = 4\alpha \sin.\ zm.x',$$

which belongs to a parabola.

And thus every section that a plane can make with a

cone, has been proved to be one or other of the curves discussed in the preceding Section.

248. From this method of deriving them, the curves have received their name of "conic sections;" but other solids exist, whose intersections with a plane are also limited to the curves in question; and, in fact, this property is characteristic of all surfaces of the second degree.

The general equation of such surfaces is,

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + K = o;$$

which, by a change in the primordial system, and provided we allow the constants k'^2, k''^2, k'''^2 to admit any values, from $-\infty$ to $+\infty$, can always be reduced to the form

$$\left(\frac{x}{k'}\right)^2 + \left(\frac{y}{k''}\right)^2 + \left(\frac{z}{k'''}\right)^2 = 1.$$

This fact will be more readily demonstrated in another place, but, in the mean time, we may examine the properties of the solids that are arranged under the equation.

The examination may be performed either by assuming values for *two* of the co-ordinates, and seeking from the equation the corresponding value of the *third*; or, by investigating the sections which the surface makes with a plane. These two modes of analysis have already been explained in Part I., art. 17, where the forms of surfaces were ascertained, either by determining the positions of individual points in them, or by seeking the intersections of the surfaces with planes.

If we adopt the first method, and assume y and z to have finite values, such that

$$\left(\frac{y}{k''}\right)^2 + \left(\frac{z}{k'''}\right)^2 < 1,$$

the corresponding values of x, supposing k'^2, k''^2, and k'''^2 positive, will be real, and less than k'; and by similar reasoning we discover that x, y and z, on the hypothesis assumed, are always real within the limits $x = \pm k'$, $y = \pm k''$, and $z = \pm k'''$; a result which indicates a surface closed on all sides.

When either of the co-efficients is imaginary, the equation may be put under the form

$$\left(\frac{x}{k'}\right)^2 + \left(\frac{y}{k''}\right)^2 - \left(\frac{z}{k'''}\right)^2 = 1,$$

where, reasoning as before, it is obvious, that whilst

$$\left(\frac{x}{k'}\right)^2 + \left(\frac{y}{k''}\right)^2 < 1,$$

the value of z will be imaginary; that z becomes real beyond this limit, and may be increased to infinity by suitably increasing x and y.

The limits of the latter are

$$x = \pm k' \sqrt{1 + \left(\frac{z}{k'''}\right)^2},$$

$$y = \pm k'' \sqrt{1 + \left(\frac{z}{k'''}\right)^2}.$$

The result, that z is always real whilst x and y exceed certain limits, indicates a certain degree of continuity in the surface; and, if we consider in connection with this fact the limits assigned to x and y, the surface will be

seen to possess an annular form, and to have as much continuity as is consistent with that figure.

When two of the co-efficients are imaginary, the general equation of surfaces of the second order becomes

$$\left(\frac{x}{k'}\right)^2 - \left(\frac{y}{k''}\right)^2 - \left(\frac{z}{k'''}\right)^2 = 1;$$

where, as y and z are imaginary whilst x'^2 is less than k'^2, and real when x'^2 exceeds that limit, portions of the surface must be detached, one commencing when $x = +k$, and the other when $x = -k$.

The surfaces represented by these three cases of the general equation are usually treated as distinct subdivisions; and are termed, respectively, the *ellipsoid*, the *hyperboloid of one sheet*, and the *hyperboloid of two sheets*. We shall now proceed to examine them separately.

249. Using for this purpose the analysis by sections, we observe that a plane parallel to the plane of the zy's, and at a distance a from the origin, will be represented by the equation

$$x = a;$$

and, combining this equation with the equation of the ellipsoid,

$$\left(\frac{x}{k'}\right)^2 + \left(\frac{y}{k''}\right)^2 + \left(\frac{z}{k'''}\right)^2 = 1,$$

we deduce,

$$\left(\frac{y}{k''}\right)^2 + \left(\frac{z}{k'''}\right)^2 = 1 - \left(\frac{a}{k'}\right)^2,$$

a result, that, compared with the general equation of lines of the second degree,

$$\left(\frac{y}{a}\right)^2+\left(\frac{z}{b}\right)^2=1,$$

gives the values

$$a=\frac{k''}{k'}\sqrt{k'^2-\alpha^2}.$$

$$b=\frac{k'''}{k'}\sqrt{k'^2-\alpha^2}.$$

Whence, it appears that whilst α is less than k' the sections in this direction are ellipses.

The sections formed by planes parallel to the planes of the zx's and the xy's, will be found by substituting for y, in the one case, and for z in the other, constant values β and γ, proceeding with the remainder of the operation as in the case that we have just solved, and where α was substituted for x.

The curves so deduced are also ellipses; and have, for the sections parallel to the plane of the zx's, the axes

$$c=\frac{k'}{k''}\sqrt{k''^2-\beta^2}$$

$$d=\frac{k'''}{k''}\sqrt{k''^2-\beta^2};$$

and, for the sections parallel to the plane of the xy's, the axes

$$e=\frac{k'}{k'''}\sqrt{k'''^2-\gamma^2}$$

$$f=\frac{k''}{k'''}\sqrt{k'''^2-\gamma^2}.$$

When α, β and γ are each zero, the planes in question pass through the origin, and the axes become

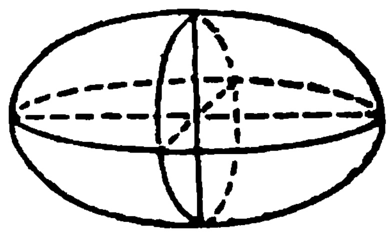
$$a'=k'',\ b'=k''',\ c'=k',\ d'=k''',\ e'=k',\ f=k''.$$

The curves, in this case, are called *principal* sections. And as we have

$$\frac{a}{b}=\frac{a'}{b'},\ \frac{c}{d}=\frac{c'}{d'},\ \frac{e}{f}=\frac{e'}{f'},$$

it is evident, that planes parallel to the co-ordinate planes, and meeting the surface, form, by their intersection with the latter, ellipses that are respectively similar to the principal sections. The directions of the axes in a principal section and its parallel, will also coincide; and as their centres lie in the same primordial axe, the form of the solid must be symmetrical on every side, fig. 239.

Fig. 239.



250. Analysing, in like manner, the equation

$$\left(\frac{x}{k'}\right)^2+\left(\frac{y}{k''}\right)^2-\left(\frac{z}{k'''}\right)^2=1,$$

we find for the equation of its sections parallel to the three co-ordinate planes

$$\left(\frac{y}{k''}\right)^2-\left(\frac{z}{k'''}\right)^2=1-\left(\frac{\alpha}{k'}\right)^2$$

$$\left(\frac{x}{k'}\right)^2-\left(\frac{z}{k'''}\right)^2=1-\left(\frac{\beta}{k''}\right)^2$$

$$\left(\frac{x}{k'}\right)^2+\left(\frac{y}{k''}\right)^2=1+\left(\frac{\gamma}{k'''}\right)^2.$$

The first of these equations indicates an hyperbola, the semi-axes of which are,

$$a=\frac{k''}{k'}\sqrt{k'^2-\alpha^2}$$

$$b=\frac{k'''}{k'}\sqrt{k'^2-\alpha^2}.$$

The second belongs also to an hyperbola, and gives for the semi-axes,

$$c = \frac{k'}{k''} \sqrt{k''^2 - \beta^2}$$

$$d = \frac{k'''}{k''} \sqrt{k''^2 - \beta^2}.$$

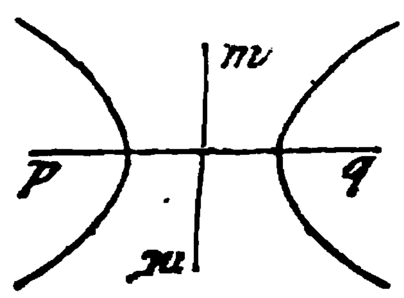
But the third equation indicates an ellipse, with the semi-axes

$$e = \frac{k'}{k'''} \sqrt{k'''^2 + \gamma^2}$$

$$f = \frac{k''}{k'''} \sqrt{k'''^2 + \gamma^2}.$$

The form of this surface can with difficulty be represented graphically, but some notion of it may be formed by considering the particular case wherein $k' = k''$; the sections parallel to the plane of the xy's will then be circles, and the surface would be generated by the revolution of two opposite hyperbolas about their minor axis mn.

Fig. 240.



When α and β are equal to k' and k'', the two first equations become

$$\left(\frac{y}{k''}\right)^2 - \left(\frac{z}{k'''}\right)^2 = o,$$

$$\left(\frac{x}{k''}\right)^2 - \left(\frac{z}{k'''}\right)^2 = o;$$

the former indicating two right lines inclined to the axe of the y's, at the angles $\tan.^{-1} \frac{k'''}{k''}$ and $-\tan.^{-1} \frac{k'''}{k''}$; and the latter, straight lines inclined at angles $\tan.^{-1} \frac{k'''}{k'}$

and $-\tan.^{-1}\dfrac{k'''}{k'}$ to the axe of the x's.

These lines are, in fact, limits, beyond which a change occurs in the form of the section, the hyperbolas within the lines turning their convexities to the plane of the xy's, and, without the lines, turning their concavities.

251. The third form of surfaces of the second degree, expressed by the equation

$$\left(\frac{x}{k'}\right)^2 - \left(\frac{y}{k''}\right)^2 - \left(\frac{z}{k''}\right)^2 = 1,$$

when examined by a similar analysis, gives, for sections parallel to the co-ordinate planes, the equations

$$\left(\frac{y}{k''}\right)^2 + \left(\frac{z}{k'''}\right)^2 = \left(\frac{\alpha}{k'}\right)^2 - 1,$$

$$\left(\frac{x}{k''}\right)^2 - \left(\frac{z}{k'''}\right)^2 = 1 + \left(\frac{\beta}{k''}\right)^2$$

$$\left(\frac{x}{k'}\right)^2 - \left(\frac{y}{k''}\right)^2 = 1 + \left(\frac{\gamma}{k'''}\right)^2.$$

The two latter indicate hyperbolas, the axes of which are, respectively,

$$c = \frac{k'}{k''}\sqrt{k''^2 + \beta^2},$$

$$d = \frac{k'''}{k''}\sqrt{k''^2 + \beta^2},$$

$$e = \frac{k'}{k'''}\sqrt{k'''^2 + \gamma^2},$$

$$f = \frac{k''}{k'''}\sqrt{k'''^2 + \gamma^2}.$$

The first equation is imaginary when $\alpha < 1$; but, when $\alpha > 1$, the section is an ellipse, having the axes

$$a = \frac{k''}{k'} \sqrt{\alpha^2 - k'^2},$$

$$b = \frac{k'''}{k'} \sqrt{\alpha^2 - k'^2};$$

and as the same result follows whether α is positive or negative, we conclude the surface to consist of two detached surfaces, placed on opposite sides of the planes of the yz's.

When k'' is equal to k''', the sections parallel to the plane of the yz's, in each of the three cases that we have considered, become circles, and the surfaces are then said to be "of revolution." The ellipsoid of revolution, or the spheroid, is formed by the revolution of an ellipse about one of its axes; if round the major axe pq, the surface is termed a "prolate" spheroid, or ellipsoid of revolution: when formed by a revolution about the minor axe, the surface is termed "oblate." The hyperboloid of revolution of one sheet is formed by the revolution of the opposite hyperbolas about the minor axe mn; and the hyperboloid of revolution of one sheet, by the revolution of the same hyperbolas about their major axe.

Fig. 241.

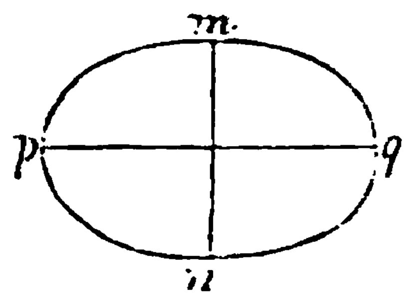
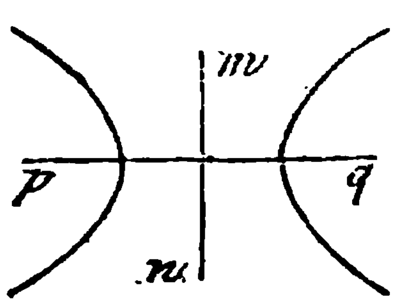


Fig. 242.



252. The three preceding cases include all those forms of surfaces of the second degree, wherein k', k'' and k''' are finite. But one or more of these axes may be infi-

nite; and the surface expressed by the given equation belongs then to a class that are known as *paraboloids*.

Assuming the infinite axes to be k'' and k''', a process similar to that used in art. 237, will reduce the equation to the form

$$ax^2 + by^2 = cz;$$

which, analysed by sections, presents the following results.

Making $z = o$, we obtain the section formed by the plane of the xy's; and as the equation then becomes

$$ax^2 + by^2 = o,$$

this section must be a point, and is, in fact, the origin of the co-ordinates. On a plane parallel to the xy's, and at a distance γ, the equation becomes

$$\frac{a}{c\gamma}\,x^2 + \frac{b}{c\gamma}\,y^2 = 1,$$

which indicates an ellipse, having the axes

$$\sqrt{\frac{c\,\gamma}{a}} \text{ and } \sqrt{\frac{c\,y}{b}}.$$

On the planes of the xz's and yz's the sections are parabolas, and their parameters are, respectively, $\frac{c}{a}$ and $\frac{c}{b}$.

The sections are also parabolas when the intersecting planes are parallel to the planes of the xz's or yz's; but as the equations then assume the form

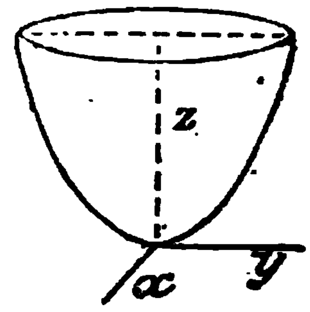
$$ax^2 = cz - b\beta^2,$$

or,

$$by^2 = cz - a\alpha^2,$$

the vertex of the figure does not lie in the plane of the yx's.

Fig. 243.

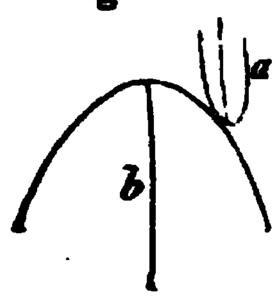


Such a surface is termed an *elliptic* paraboloid, and has the form of a parabolic cup, the sections of which, at right angles to the axis, are ellipses.

When b is negative, the ellipse of the preceding case becomes an hyperbola; and the parabolic sections formed by planes parallel to the yz's are situated differently from the parabolic sections formed by planes that are parallel to the xz's; the one set being situated "above" the xy's, and the other "below."

This surface is termed the *hyperbolic* paraboloid.

Fig. 244.



Its form is not readily exhibited by a diagram; but an accurate idea of it may be obtained by regarding the surface as traced by a parabola, restricted to move on a curve of the same kind.

The axes of these parabolas are supposed parallel, and their planes at right angles; and the motion must be such as to preserve this arrangement, and maintain the vertex of a in the periphery of b.

When a and b are equal, the elliptic paraboloid becomes a solid of revolution, but a like remark does not apply to the hyperbolic paraboloid.

PRELIMINARY REFLECTIONS TO SECTIONS V., VI., VII. AND VIII.

The principle of arrangement that assigns the place of a curve by the degree of its equation, applies only to the curves the equations of which are algebraic. But the idea of a curve, or of a mathematical line of any kind, is derived, as explained in Part I., from the intersection of two surfaces; and thus every principle of arrangement that applies to the latter species of quantity, applies also to curve lines.

INQUIRY SUGGESTED BY THESE REFLECTIONS.

Of lines considered as the intersections of surfaces. Most commodious method of arranging curve surfaces.

SECTION V.

OF LINES CONSIDERED AS THE INTERSECTIONS OF SURFACES.

Remarks on the arrangement of lines and surfaces—lines regarded as the intersections of surfaces—of the helix—of the spiral described by the sun—of curves arranged as the locii of points subjected to known motions.

253. Regarding all geometrical investigations as having for their subject matter the relations of points in space, we have adopted a principle of arrangement, that, proceeding on the number of the latter, classes lines and surfaces as figures involving the positions of an infinite number of points. Viewed in this light, the analysis of position and quantity appears simple, methodical and general; but the vast extent of the inquiry renders this method still insufficient, and requires that we should unite with it principles of subdivision. The labours of eminent geometricians have been directed to this object, but the arbitrary nature of all subordinate classifications has prevented their agreeing on any uniform and complete system.

The degree of the equation, and the facility with which the curves could be represented graphically, have been employed as subordinate principles of arrangement; but it is obvious that both are incomplete.

Chap. I. Of lines and surfaces.

Art. 253. Remarks on the arrangement of lines and surfaces.

The graphical arrangement of lines is used when we divide them into "plane curves," and curves of "double curvature," or, in other words, into curves every point of which lies in a plane, and curves not so restricted; but this method, although highly useful, has the disadvantage of restricting, to a greater extent than other methods, the generality of the conclusions obtained respecting lines, and of destroying the unity which their theory would otherwise possess.

This remark will be more obvious if we recur to what is said in Part I., art. 3, concerning the notion we entertain of lines: our first ideas of form, it is there shown, are derived from an acquaintance with solid bodies; whilst the more abstract notions of surfaces, and of lines the boundaries of surfaces, are obtained by neglecting, in the complex idea of a solid, certain of its parts.

These views will lead immediately to the several conditions imposed on the co-ordinates of a point, according as it is completely given, is restricted to lie within a *line*, a surface, or a solid.

The parts of the latter filling the portion of space enclosed within its surface, the co-ordinates of a point in the solid will be merely subject to the imperfect conditions that have reference to the limits within which they must be taken.

These limits are the co-ordinates of points in the surface of the solid; and hence, if we arrange the varieties of form according to the order in which the ideas of them are acquired, the surface will be the first wherein the condition restricting the points will be sufficiently definite to lead to an equation.

Assuming for the latter

$$\phi\,(x, y, z) = o,$$

we readily perceive that x, y and z are susceptible of such continuous values as would belong to a surface, and that every equation between three variables admits of a similar interpretation.

Assuming the co-ordinates x, y, z to be restricted by two equations,

$$\phi(x, y, z) = o, \quad \psi(x, y, z) = o,$$

we may regard either of these, when taken separately, as indicating a surface. And as the co-ordinates of all points in the surface indicated by the first equation will fulfil that equation, and as a similar remark extends to the second surface, it will follow that when both equations are simultaneous, the co-ordinates belong to both surfaces, or to the line which is their intersection.

Two equations between these co-ordinates are thus shown to indicate a line, either curved or straight; and as the reasoning may be extended to the intersection of three surfaces, or two lines, it will also follow that three equations indicate a definite number of points; a result consistent with the known rules of analysis, which restrict the roots of such equations, or the values of x, y and z, to be equal in number to the product of the degrees of the equations.

The principles here developed, render evident the two chief methods that may be employed in the arrangement of lines and surfaces.

According to the first, the theory of lines is completed before the theory of surfaces is commenced.

According to the second, the proceeding is reversed, and the theory of surfaces is regarded as leading to the theory of lines.

The extent of the subject, and the imperfections of

analysis, have led mathematicians to reject neither of these principles; but, to employ the first in treating of plane curves and of surfaces, and the last in discussing curves of double curvature.

254. Following, whilst treating of these curves, the arrangement in question, we shall regard all lines as the intersections of two surfaces, or as represented analytically, art. 253, by two simultaneous equations.

A just combination of the latter, serving to eliminate one of the variables, will enable us to reduce the equations to others, in either of which a variable shall be wanting; and the equations, when so arranged, may be regarded, art. 231, as representing the "projections" of the line on two of the co-ordinate planes.

The truth of this last assertion will require some further development, and, for that purpose, let us assume the line as represented algebraically by the equations

$$\phi\,(x, y, z) = o,$$
$$\psi\,(x, y, z) = o.$$

Eliminating y from the first, and x from the second, they become

$$\mathrm{F}\,(x, z) = o,$$
$$f\,(y, z) = o.$$

The first of these equations, taken by itself, represents a surface that passes through the given line; and if we unite it with a simple equation in x and z, or, in other words, with the equation of a plane parallel to the axe of the y's, we shall have for the intersection of the plane and surface, two equations in x and z.

Uniting these, we obtain

$$x = \text{cons}.$$
$$z = \text{cons}.,$$

which, belonging separately to planes parallel, respectively, to those of the yz's and xy's, indicate a line parallel to the axe of the y's.

The surface, the equation of which is $F(x, z) = o$, is formed, therefore, by right lines that pass through the given curve, and in directions parallel to the axe of the y's.

Now such lines are precisely the perpendiculars let fall from the several points of the given curve to the plane of the xz's; and the intersection of the latter plane with the surface represented by the equation

$$F(x, z) = 0,$$

will, therefore, be the projection of the curve in question upon the co-ordinate plane above mentioned.

The equation of the plane of the xz's is $y = 0$, and if this is combined with the equation of the surface, the result will indicate the projection of which we are in search. But the equation of the surface not containing y, is not altered by the values assigned that quantity, and we may regard

$$F(x, z) = 0$$

as being, itself, the equation of the projection in question.

The result of this investigation confirms the proposition of art. 231, and establishes in a more general manner that two equations

$$F(x, z) = 0,$$
$$f(y, z) = 0,$$

which taken simultaneously represent a line, express when taken separately the projections of the line upon the planes of the xz's and the yz's.

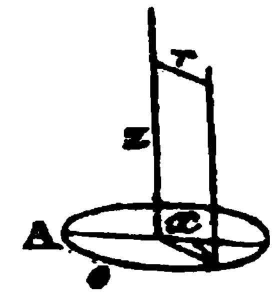
A similar remark applies to the equation obtained by

eliminating z; and as this equation is an immediate result from the two preceding, we conclude, in conformity with the theory developed in the second Chapter of Part II., that either of the projections of a line is dependent upon the two remaining projections.

255. The theory here given of lines of double curvature, will be sufficiently illustrated by one or two examples.

The first that we shall employ for that purpose is the *helix*, a curve that has already occupied our attention in Part I., where it is delineated in figs. 4, 5 and 6.

Fig. 245.



To investigate the equations of the helix, we may consider it as the intersection of a cylindric surface with a surface which we have yet to describe, and which would be generated by a line r, subjected to two uniform motions; one of rotation about z, by which r described an angle θ, and the other a motion which caused the line r to ascend upon z, at the same time that it remained constantly parallel to the plane of the xy's.

The equation of the cylinder, art. 244, is

$$x^2 + y^2 = a^2,$$

whilst the equation of the second surface, as appears from the mode of its generation, will be

$$z = m\theta.$$

And these two equations, taken conjointly, are, therefore, equations of the curve sought: but as they contain four co-ordinates, it will be convenient to substitute for θ its value in terms of x and y.

Now,

$$\cos.\ \theta = \frac{x}{a}$$

$$\sin.\ \theta = \frac{y}{a},$$

whence, substituting in the equation of the second surface these values of θ, we deduce for the equations of the helix,

$$z = m \cos.^{-1} \frac{x}{a}$$

$$z = m \sin.^{-1} \frac{y}{a}.$$

The first, according to what has been remarked in the preceding article, representing the projection of the curve on the plane of the xz's; and the second its equation upon the plane of the zy's.

The discussion of these equations will be attended with little difficulty.

The form of them indicates that x and y can only be taken between the limits plus and minus a; and, also, that for any assigned values of these co-ordinates, z has an infinity of values, which may all be included in either of the formulæ,

$$z = m \left(n\pi + \cos.^{-1} \frac{x}{a}\right)$$

$$z = m \left(n\pi + \sin.^{-1} \frac{y}{a}\right);$$

where the second terms in the right hand members are the least values of $\cos.^{-1} \frac{x}{a}$, $\sin.^{-1} \frac{y}{a}$; and where n is a whole number.

From these equations the curve appears to consist of an infinite number of branches, forming so many spiral

coils, resembling those in the thread of a screw: they are equal, and similarly disposed, and are separated by the interval $2m\pi$.

The section by a plane parallel to one of the co-ordinate planes, will be obtained by substituting a constant quantity for x, y or z.

The last substitution, if we assume $z = \gamma$, gives,

$$\frac{x}{a} = \pm \cos. \frac{\gamma}{m}$$

$$\frac{y}{a} = \pm \sin. \frac{\gamma}{m};$$

whence the intersection is a point.

The two first substitutions will lead to results somewhat different from this, but agreeing with each other, and which will be sufficiently explained by examining either substitution.

Making choice of the first, and assuming with that view $x = \alpha$, we obtain

$$z = m\left(n\pi + \cos.^{-1} \frac{\alpha}{a}\right),$$

which indicates two infinite series of points, arranged in parallel lines, and having the consecutive points in each series separated by the common interval $2m\pi$.

256. The second example whereby we shall illustrate the theory that has occupied our attention in the present section, is remarkable, as being the curve in which the sun would appear to move, were the motion of the luminary regular, and the "ecliptic," or the annual path of the sun, a circle. It is best conceived by supposing a point to move uniformly in a great circle of the sphere, whilst the sphere itself revolves with an equable motion round one of its diameters.

This diameter is called the "axis of motion," and the great circle to which it is perpendicular is termed the "equator."

Now let us suppose this sphere to be placed within a second sphere, concentric with the first; and having (the first sphere is regarded as a solid, and the second as a spherical shell) the exterior surface of the former in contact with the interior surface of the latter. The motion of the point A, arising from the rotation of the interior sphere, and the proper motion of the point, may then be rendered sensible, by supposing A to leave a trace of its course upon the interior surface of the fixed, or outer sphere.

It is the curve so traced that we are now to examine.

Assuming any point O' in the equator of the interior sphere to be coincident with a known point O in the sphere which is at rest, the rotation of the former will cause the points O' and O to separate, and the distance between them, estimated on the equator, is a measure of the angle through which the sphere has revolved.

Let this distance be ϕ.

The hypothesis supposes that whilst the sphere is revolving, a point A, placed on its surface, and partaking of the motion of rotation, has also a motion peculiar to itself, and in virtue of which it moves upon an oblique circle of the sphere; describing by this proper motion an arc ψ, whilst it is carried by the motion of revolution through the arc ϕ.

As both motions are uniform, the arcs ψ and ϕ must be proportional, and we have

$$m\psi = \phi.$$

This simple expression is an equation to the curve

sought; and as it is expressed in terms of arcs, ψ and ϕ, that are described upon the surface of a sphere, we may regard it as equivalent to two equations between three co-ordinates, or, in other words, as completely defining the curve in question.

The arcs ψ and ϕ belong to a system of oblique spherical co-ordinates, and may be easily transformed by the rules of spherical trigonometry, into other arcs that refer to a system which is rectangular; the simplicity of the equation, however, would be lost by such transformation, nor is the latter necessary to an analysis of the nature and properties of the curve.

To effect this analysis we may proceed as follows.

Putting α for the invariable angle that measures the inclination of the "ecliptic," or oblique circle, to the equator, and denoting by y the distance of the point A from the latter, we have

$$\sin. y = \sin. \alpha \sin. \psi,$$

whence it appears that y increases from 0 to $\sin. \alpha$, as ψ increased from 0 to $\frac{1}{2}\pi$. When ψ exceeds this last value, y decreases, until when ψ is π the value of y again becomes zero. For values of ψ exceeding π, y is negative, and it is obvious that values of y, corresponding to ψ, and $\pi + \psi$, differ only in their sign.

Now assuming m to be a large positive number, and ψ', ψ'', ψ''', &c. to be values of ψ corresponding to ϕ, $2\pi + \phi$, $4\pi + \phi$, &c., the equation

$$m\psi = \phi$$

gives

$$\psi' = \frac{\phi}{m}$$

$$\psi'' = \frac{2\pi + \phi}{m}$$

$$\psi''' = \frac{4\pi + \phi}{m}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\psi''^{\,n+1} = \frac{2n\pi + \phi}{m},$$

whence,

$$\sin. y' = \sin. a \sin. \frac{\phi}{m}$$

$$\sin. y'' = \sin. a \sin. \frac{2\pi + \phi}{m}$$

$$\sin. y''' = \sin. a \sin. \frac{4\pi + \phi}{m}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\sin. y''^{\cdots n+1} = \sin. a \sin. \frac{2n\pi + \phi}{m}$$

Whilst $2n\pi + \phi$ is less than $\frac{1}{2} m\pi$, the series y', y'', y''', &c. is increasing, and as these values of y are the ordinates of the points where a secondary to the equator would be met by the curve, we conclude the latter to be a spiral consisting of folds that rise at each convolution higher upon the sphere.

The knowledge we possess of the relations between angles and the ratios of the type of closed figures, will enable us to examine more closely the spiral here analysed. For as the ratio

$$\frac{\sin. a}{a}$$

is known from those relations to decrease to zero, with the increase of a, it follows that constant additions to angles of different magnitudes do not equally increase the sines of the angles; the addition to the greater angle not

only producing the least effect upon the sine, but, finally, when the angle is nearly $\frac{1}{2}\pi$, producing in the sine an increase indefinitely less than the increase of the angle.

Bearing this fact in mind, it will be observed that whilst $2n\pi + \phi < \frac{1}{2}m\pi$, the differences of the consecutive terms in the series y', y'', y''', &c. continue to decrease, which proves the folds of the spiral not to be equally distant, but to approach as they ascend upon the sphere, until, when $2n\pi + \phi = \frac{1}{2}m\pi$, the superior fold meets that which is next inferior, and the spiral thence continues to descend towards the equator by folds, similar to those of the ascending branch.

The identity of the portions of the oblique circle that are situated above and below the equator, will alone demonstrate, that when $2n\pi + \phi = m\pi$, the point A enters upon a branch of the spiral identical with that which it has left, but placed below the equator, in a position agreeing with the position of the latter branch above the equator.

If $\frac{1}{m}$ is a whole number, the point A, after describing the oblique circle once, will arrive at the point with which it was at first coincident, and the spiral will then consist merely of the double branches made known by the preceding analysis.

When $\frac{1}{m}$ is a fraction, $\frac{p}{q}$, in its lowest terms, the spiral will contain as many pairs of double branches as there are units in p. And, lastly, when $\frac{1}{m}$ is an endless decimal, the number of double branches will be infinite, or, in other words, the point A will never again become coincident with the point O.

Sect. V. Of lines considered as the intersections of surfaces.
Art. 257. Of curves arranged as the loci of points subjected to known motions.

257. The last problem suggests a method of classing curves that differs from those we have hitherto examined, but of which the limits of the work will only allow a passing notice. This method regards curves as described by a moving point, and classes them according to the known motions that affect the latter.

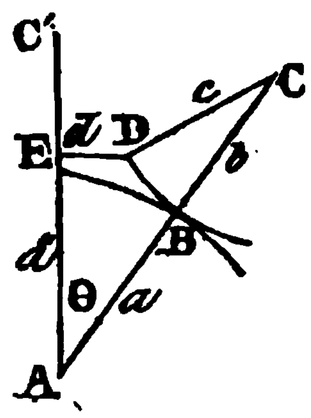
The circle is an obvious example of this kind, and many others might be selected, but we shall prefer giving a detailed analysis of a single class of curves, the *epicycloids*, that readily admit the arrangement in question.

The epicycloid is the curve which a point in the circumference of a circle describes, when the latter revolves on the circumference of a second circle.

The latter is known as the *base*, and the circle which revolves upon it is termed the *generating* circle.

Now as every point of the generating circle is successively in contact with every point of the base, the arc of the latter, intercepted between any two points of contact, must be equal to the corresponding arc described by the revolving circle; and the angles measured by these arcs will be reciprocally as the radii. If, therefore, A and C′ are the positions of the centres when the motion commences, and A and C their positions at any other time, we have

Fig. 246.



$$c = b,$$

$$(bc) = \frac{a}{b}\,(aa''')$$

$$(ca''') = \frac{a}{b}(aa''') + (aa''')$$

$$= \frac{a+b}{b}\,(aa''');$$

and putting x and y in place of a''' and d, and θ in place of aa''', the equations of closed figures give

$$x = (a + b) \cos. \theta - b \cos. \frac{a+b}{b} \theta$$

$$y = (a + b) \sin. \theta - b \sin. \frac{a+b}{b} \theta;$$

which are the equations of the curve in question.

If the describing point is not situated in the circumference of the describing circle, but is merely restricted to be a fixed point in the plane of the latter, the radius c is not equal to b, and the equations become

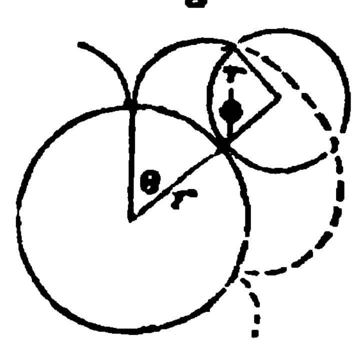
$$x = (a + b) \cos. \theta - c \cos. \frac{a+b}{b} \theta,$$

$$y = (a + b) \sin. \theta - c \sin. \frac{a+b}{b} \theta.$$

This class of curves are termed *epitrochoids*.

The form of these curves will be most readily discussed by regarding the centre of the generating circle as a movable origin. The co-ordinates relative to the fixed origin will then be the distance r from the centre of the fixed, to the centre of the revolving circle, and the angle θ through which this radius has revolved; whilst, relative to the movable origin, they will be the radius r' and the angle ϕ through which it has revolved.

Fig. 247.



The equations of the curve, with these conventions, will be simply r and r' equal to constants, and

$$\phi = \frac{r - r'}{r'} \theta.$$

When the circle has made a com-

plete revolution, or $\phi = 2\pi$, θ will be equal to

$$\frac{2r'\pi}{r-r'};$$

and hence, if the circumference of the base is divided into arcs equal to the latter quantity, the branches of the curve described upon these arcs will be similar and equal.

If $\frac{r'}{r-r'}$ is commensurate with unity, the number of branches will be finite; but when $\frac{r'}{r-r'}$, is an incommensurable number, the curve never returns into itself.

When the radius of the base is infinite, the angle θ is infinitely small, and its cosine may be considered as equal to unity, and its sine as equal to the arc which measures θ; with these substitutions the values of x and y become

$$x = (a + b) - c \cos. \phi,$$
$$y = (a + b)\,\theta - c \sin. \phi;$$

which, shifting the origin in the direction of the x's by the quantity a, become

$$x = b - c \cos. \phi,$$
$$y = b\,(\phi - c \sin. \phi,$$

equations that belong to the *trochoid*, a name given to that variety of the general curve in which the base is a straight line. If c is equal to b, or the describing point lies in the circumference of the generating circle, the trochoid becomes the *cycloid*, and the equations are

$$x = b\,(1 - \cos. \phi)$$
$$y = b\,(\phi - \sin. \phi).$$

Equations that are in many cases more convenient, are obtained by shifting the origin to the summit of one of

the cycloidal branches: the summit of the first branch corresponds to $\phi = \pi$, which gives $x = 2b$, and $y = b\pi$; and we shall have for the equations of transformation

$$x = 2b - x',$$
$$y = y' + b\pi,$$
$$\phi = \pi + \phi';$$

with these substitutions, the equations become

$$x' = b\,(1 - \cos.\,\phi'),$$
$$y' = b\,(\phi + \sin.\,\phi).$$

SECTION VI.

METHOD OF ARRANGING LINES AND SURFACES BY PARAMETERS.

A parameter, in the arrangement of mathematical quantities, is a variable by means of which we pass from a subdivision to a division—parameters may be measured as co-ordinates—of simple and complex systems of lines—transformation of parameters—dependent parameters are co-ordinates of points wherein the lines or surfaces of the system intersect a line or surface that does not belong to it—two equations between three co-ordinates and a parameter, indicate a line lying in a given surface—two equations between three co-ordinates and two parameters, indicate a system of curves that do not lie in a surface.

258. We have given, in Sections II. and IV., a detailed analysis of certain classes of curved surfaces, and have touched, in the preceding Section, upon the principles of subdivision required in arranging these varieties of form.

But even the brief consideration that was there given to the subject, sufficed to show the imperfect state in which the arrangement of surfaces has been permitted to remain.

Lines, we have seen, are classed as the intersections of surfaces; whilst surfaces themselves, when their equations are complex, are arranged with reference to the lines they contain.

This double process cannot be avoided in the present state of the science; and we shall now, therefore, proceed to consider lines under the aspect they assume when regarded as composing a surface.

Lines, according to this view, must be esteemed as existing, not separately, but in groups; and equations must be obtained that will apply to a whole group at once.

A little attention to the theory of lines and surfaces will render this comparatively easy. We saw, in the preceding Section, that an equation between *three* variables represents a surface, and that two equations between three variables represents a line.

Let us now examine the geometrical relations that would be indicated if, instead of three, we had *four* variables; assuming them, as a first case, to be connected by the two equations,

$$\varphi\,(x, y, z, p) = 0,$$
$$\psi\,(x, y, z, p) = 0.$$

If we regard p as constant, the equations will belong to a line, but to a line that varies with every value assigned to p.

For two equations between these co-ordinates, therefore, to represent a single line, it is necessary that every

quantity which enters those equations, the co-ordinates excepted, should be definite, and incapable of taking more than one value; in other words, the equations must contain merely numbers, and the three letters which denote the co-ordinates; since literal co-efficients, admitting of any values that we please to give them, render the equations applicable to an infinity of lines.

In the case we are considering, the two equations contain merely numbers, and the four letters x, y, z and p: whence, eliminating this last, a third equation,

$$\zeta(x, y, z) = 0,$$

arises, that, containing only x, y and z, indicates a surface; and which, as independent of the particular value given to p, indicates the surface wherein all the lines in question are situated.

The quantity p is here viewed both as a constant and a variable; we view it as a constant, whilst examining an individual line, and as a variable when passing from one line to another.

Such, as the name implies, is the idea attached to every arbitrary constant; the quantity is constant under a certain point of view, and variable when that view is no longer the same.

The distinction between the views under which the quantity is regarded as variable or constant, is more readily perceived in geometry than in other branches of mathematics, but in all, it supposes the problems, or the things treated of, to have been classed into divisions and subdivisions; it is in reference to the last that the arbitrary constants are regarded as assigned; in reference to the latter that they lose their character of constants, and

assume that of arbitrary, and, therefore, variable quantities.

The co-ordinates x, y and z, for example, in the preceding equation, refer, when p is "constant," to points arranged in a certain line; but when p is "variable," they are no longer restricted to a single line, but have reference to points situated any where in that group among which the individual line in question is classed.

The group of lines forms, in this case, a division, under which certain points in space are arranged; the individual line is a subdivision, distinguishing certain of the points in the group from the remainder; and the quantities that enable us to pass from one to the other, from the subdivision to its proximate division, are termed, in mathematics, *parameters*, and act a very important part in the analysis of form and magnitude.

The theory of such quantities will be most readily illustrated when restricted, in the first place, to plane curves.

The circle, for example, when represented by the equation

$$y^2 + x^2 = r^2,$$

is expressed in terms of two co-ordinates, x and y, and a parameter r. And we immediately perceive, that, whilst r is constant, the equation belongs merely to a particular circle, and the co-ordinates to the points that lie in it; but that when r varies, the meaning of the equation becomes more extended, and the points, of which x and y are co-ordinates, are no longer restricted to the circle in question, but may belong to any circle having the same centre.

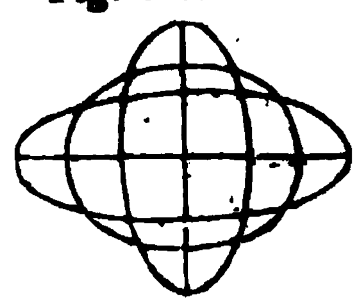
The equation of this *system* of concentric circles is,

we observe, equivalent to the equation of the plane wherein they lie; for as every point of that plane must lie in some one of the circles in question, the equation which is common to all the latter must apply to every point of the plane. Such, however, is not always the case; the lines included in the system may cross each other, and thus comprehend the same point more than once. The equation, for example,

$$y^2 + p^4x^2 = p^2,$$

belongs to a set of ellipses that are concentric without being similar, and where each pair of ellipses have, therefore, four points in common.

Fig. 248.

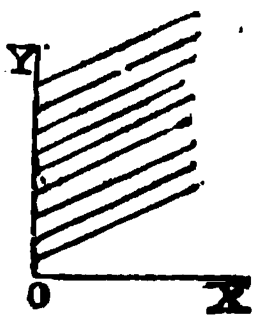


The parameter p, in this equation, may be measured upon the axe of the y's; for, making x equal to zero, and denoting the corresponding value of y by y' we obtain

$$p = \pm y'.$$

259. This remark will lead us to perceive that a parameter of a system of plane curves may be regarded, when only one parameter enters the equation, as an ordinate corresponding to some constant value of the remaining co-ordinate. In the system of straight lines, for example, expressed by the equation

Fig. 249.



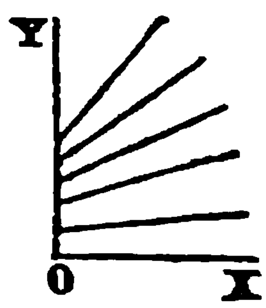
$$y = ax + p,$$

it is evident, that, whilst p is arbitrary, the equation belongs, not to one straight but to an infinite number of straight lines; and that p, the parameter by which they are distinguished, is equal

to the value y', which y takes when x becomes zero: and it is further evident, that a knowledge of y', or of the point where the line cuts the axe of the y's, together with a knowledge of a, or of the tangent of the inclination with the axe of the x's, are data that suffice to determine the line.

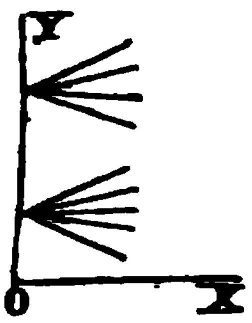
260. As a, in this example, is supposed constant, the lines are parallel; but a being a quantity to which we may assign any numerical value, it may also be regarded as variable; and two cases will then present themselves, the case, namely, in which a and p are regarded as independent, and the case in which a is regarded as a function of p.

Fig. 250.



These cases indicate, that systems of lines may be arranged under the two classes of *simple* and *complex* systems, according as the individual lines which they contain are assigned by means of one, or many points.

Fig. 251.



An example that falls under the first of these classes is exhibited in figure 250, where every line is assigned by the single point wherein it meets the axe of the y's: whilst in figure 251, which represents a complex system, the individual lines are not completely distinguished by the point wherein they meet the axe of the y's, but require that we should also have given some other point through which they pass, the point, for example, wherein they meet the axe of the x's. This last remark will be more readily seen by considering that every point in the axe of the y's, or, which is the same thing, every value of p, will belong to each of an infinite number of

lines, diverging from this point in all directions, and only distinguished apart by the value of a.

When a is a function of p, the direction of the line depends upon the point in which it meets the axe of the y's; and as only one line then belongs to each point of that axe, the system is simple, and is assigned by the two equations

$$y = ax + p, \qquad a = \mathrm{F}p.$$

The quantities a and p, of these equations, are merely ordinates of those known points through which the several lines in the system are supposed to pass, and will, therefore, be subject to all the transformations that co-ordinates admit: the angular co-ordinate a, for example, may, by such transformations, be removed; and we may substitute in its place, as a second parameter, p', the value of x that corresponds to $y = 0$.

261. The most obvious method of reducing the equation to the form necessary for this purpose is, to make x or y equal to zero, and to eliminate p between the equation so obtained, and the original equation in x and y. This simple process is evidently applicable to all cases, and that without regard to the number of parameters which the equation may contain. Thus, assuming as an example of the kind supposed the complex system expressed by

$$\phi\,(x, y, p, p') = 0,$$

and denoting by y° and x° those values of x and y that correspond respectively with $x = 0$ and $y = 0$, we have

$$\left.\begin{aligned} &\phi\,(y^\circ, p, p') = 0 \\ &\phi\,(x^\circ, p, p') = 0 \end{aligned}\right\} \quad \ldots\ldots\ \alpha$$

which, with the original equation, suffice to eliminate p and p'.

Other modes of elimination, that may be used in conjunction with a transformation of the co-ordinates, will readily suggest themselves.

Let there be given, for example, the equation

$$x^2 - 2py - a^2 - p^2 = 0;$$

by assuming

$$x = x' - a,$$

we have

$$x'^2 - 2ax - 2py - p^2 = 0;$$

and again, assuming

$$p = my,$$

there results, when $x = 0$,

$$(2m - m^2)\, y^0 = 0,$$

which is satisfied by making $m = z$; and whence is derived the transformed equation

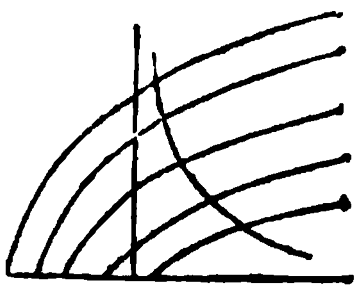
$$x^2 - 2ax - 4y^0y - 4y^{0z} = 0.$$

262. When two parameters are found in the equation of a system of lines, we have already seen, art. 259 and 260, that we may either measure them upon the axes, or regard them as co-ordinates of a line that does not enter into the system: the discussion which this subject has already obtained, in the preceding articles, has, indeed, anticipated this remark, as well as others immediately suggested by the case before us; and in reference to which it will be proper to illustrate the subject by a second example:

$$y^2 = px + p',$$
$$pp' = 1.$$

The first of these equations will indicate a system of parabolas; whilst the relation expressed by the second between p and p', assigns to those quantities, when measured from the origin O, and in directions parallel to the axes, the values assigned to the co-ordinates of points that lie in a rectangular hyperbola having the axes themselves for its asymptotes.

Fig. 252.



Assuming any quantity α, as the value of p, the equation $pp' = 1$ determines the corresponding value, β, of p'; and these quantities, measured in the same manner as the values of x and y, are co-ordinates of the point wherein the hyperbola, or, in other words, "the line of the parameters," intersects that line in the system which is assigned by the parameters α and β.

Generalizing these remarks, and extending them to lines not restricted to lie in one plane, we perceive that parameters class and arrange lines and surfaces in the same manner as ordinates class and arrange points; and that equations among such quantities must express relations common to many systems of lines and surfaces, as equations among co-ordinates expressed relations common to many points.

263. Thus, two equations between three co-ordinates and a parameter, will indicate a system of lines of double curvature, lying in a given surface. The truth of this remark will appear sufficiently obvious, if we consider, that, by eliminating p from one of the equations, and x from the other, they may be put under the form

$$\phi(x, y, z) = 0,$$
$$\psi(y, z, p) = 0,$$

where the upper equation, not containing p, will belong to a single surface; whilst the lower equation, containing p, will belong to an infinity of surfaces, that, by their intersections with the surface first mentioned, form as many lines of double curvature lying in the latter.

Assuming, in the equations above given, $x = 0$, they become,

$$\begin{aligned} \phi(y^0, z^0) &= 0, \\ \psi(y^0, z^0, p) &= 0; \end{aligned}$$

whence, by elimination, we find for p a value that may be denoted, algebraically, by

$$p = \mathrm{F}(y^0, z^0);$$

and, substituting this value in the preceding equations, there results, for the equations of the lines in question,

$$\begin{aligned} \phi(x, y, z) &= 0, \\ \psi(y, z, y^0, x^0) &= 0, \\ \phi(y^0, z^0) &= 0, \end{aligned}$$

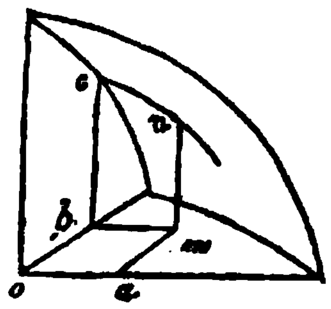
which express the relations of the lines Oa, am, an, and Ob, bc, fig. 253; the three first denoting the co-ordinates of a point n in the line cn, and the two last the parameters of that line, or the co-ordinates of the points wherein it meets the plane of the yz's.

264. Extending the inquiry, we observe, in like manner, that two equations between three co-ordinates and two parameters, represent a system of curves of double curvature that do not lie in one surface. That such is the fact, may be established by the method used in the preceding article to establish the contrary; for, eliminating the parameters, and substituting, as in the case preceding, their values in terms of y^0 and z^0; we may now

argue that since these values remain indeterminate, the intersections of the lines with the plane of the yz's will not be confined to lie in a curve, as must have been the case had the lines themselves been confined to lie in a surface.

The disposition of the lines will be more readily understood by a reference to the diagram, fig. 253, where it will be observed that if y^0 is assumed equal to Ob, we are yet uncertain to which of these lines this equation belongs, until we have also assumed bc.

Fig. 253.



These two parameters, Ob and bc assign the curve, and thus distinguish the points forming it as constituting a class distinct from all other points in space: but although the class is assigned, the individual point is yet undetermined, and requires that we should know the co-ordinates Oa, am and mn, or the co-ordinates that are distinguished by the letters x, y and z.

As an example of this kind, let there be given the equations

$$\left.\begin{aligned} x^2 + a^2y^2 + a^2z^2 &= p \\ z - p'y &= 0 \end{aligned}\right\} \quad \ldots\ldots \alpha$$

Eliminating, by means of y^0 and z^0, the parameters p and p', we obtain,

$$\left.\begin{aligned} x^2 + a\,(y^{02} - y^{02}) + a\,(z^2 - z^{02}) &= 0 \\ yz^0 - zy^0 &= 0 \end{aligned}\right\} \quad \ldots \beta$$

where, assuming $x = 0$, y and z become, as they ought, y^0 and z^0.

Substituting, in the first of these equations, the value of z obtained from the second, it becomes

$$x^2 + a^2\left(1 + \left(\frac{z^0}{y^0}\right)^2\right) y^2 = a^2\left(1 + \left(\frac{z^0}{y^0}\right)^2\right) y^{02} \ldots . \gamma$$

The last of the equations, β, is that of a plane passing through the axe of the x's, and inclined to the plane of the xy's, at the angle $\tan.^{-1} \frac{z^0}{y^0}$; whence, denoting by y' the ordinate measured parallel to the intersection of this plane with that of the zy's, we have, putting $\tan.^{-1} \frac{z^0}{y^0} = \theta$,

$$y \sec. \theta = y'$$

and the equation γ becomes,

$$x^2 + a^2 y'^2 = a^2 y^{0\prime 2},$$

which indicates an ellipse, having its centre at the origin, and its axes equal to $ay^{0\prime}$ and $y^{0\prime}$.

These ellipses are, manifestly, not only concentric, but similar, and their arrangement will be understood by attending to the equations α.

Of these equations, the first represents an infinite number of concentric and similar ellipsoids; whilst the second belongs, as we have before remarked, to a plane passing through the axe of the x's, or the axe of the ellipsoids represented by the first equation.

The intersections of the plane with the concentric ellipsoidal surfaces, will be represented by the two equations regarded as simultaneous, and will form the system of concentric and similar ellipses that resulted from the preceding analysis.

The parameters y^0 and z^0 are here independent; but, assuming an equation

$$y^0 = Fz^0$$

to exist between them, they lose this independency, and the case agrees with the preceding; the form of the function F, however, must be known, for, should it be arbitrary, it is manifest that no part of the generality of the problem is lost.

265. From this theory of parameters it will be seen, that equations between variables admit of more than one geometrical signification. The case, for example, where a single equation exists between three variables, may be understood as representing a surface, or an infinite system of lines, according as we consider the variables as three co-ordinates, or, as two co-ordinates and a parameter. When more than three variables remain independent, we must either vary the method hitherto adopted in measuring the co-ordinates of a point, and fix the position of the latter by the parts of an open figure having more than three sides, or, adhering to that method, we must regard as parameters all the variables but three.

A similar remark applies in the case where all the points are restricted to lie in a plane; but, assuming the equations to contain n variables, the number of parameters will be $n - 2$.

The equations

$$y = (u + v) \cos. \theta - (v - a) \cos. \frac{u + v}{v} \theta,$$

Chap. I. Of lines and surfaces.

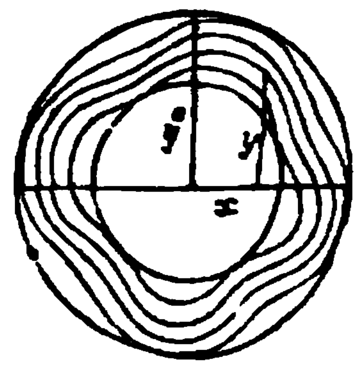
Art. 265. One equation between three variables may represent either a surface or a system of curves.

$$x = (u + v) \sin. \theta - (v - a) \sin. \frac{u + v}{v} \theta,$$

$$\gamma^2 = (u - \alpha)^2 + (v - \beta)^2,$$

where x, y, u, v, θ are regarded as variable, and a, α, β and γ as constant, may be taken as an example of this kind.

Fig. 254.



Assuming x and y to be co-ordinates, u a parameter, and θ and v dependent variables, that are to be eliminated, before the curve can be expressed by a single equation, we perceive, art. 257, the two first equations to belong to a system of epitrochoids; whilst the last assures us that the parameter u can neither exceed $\alpha + \gamma$ nor be less than $\alpha - \gamma$; and that v has, in like manner, the limits $\beta + \gamma$ and $\beta - \gamma$. When θ is zero, x is also zero, and y, which is then written y^0, becomes equal to $u + a$.

The values of y^0 are confined within the limits $a + \alpha + \gamma$, and $a + \alpha - \gamma$; whence, assuming this quantity as a parameter, and measuring it in the usual way, the equations will be seen to represent a system of epitrochoids, also confined within limits. The extent of the latter will be gathered from art. 257, whence the inferior limit of each epitrochoid appears to be a circle, having the radius $u + a$, and the superior limit a circle, with the radius $u + 2v - a$.

For the whole system, these limits become

$$a + \alpha - \gamma,$$

and,

$$-a + 2\beta + 4\sqrt{\frac{\gamma}{5}},$$

and, describing with these radii two concentric circles, fig. 254, they form boundaries within which the system of epitrochoids is confined.

SECTION VII.

ARRANGEMENT OF SURFACES BY THE LINES THEY CONTAIN.

Of the plane, regarded as a system of straight lines—of the generatrix and directrix of a surface, regarded as formed by motion—of cylindric surfaces—of conical surfaces—of surfaces of revolution—of surfaces of single curvature—of developable surfaces—of spiral surfaces.

266. The theory of parameters, developed in the preceding Section, will enable us to apply to surfaces the method of arrangement proposed in article 253.

According to this principle, surfaces would be classed by the lines they contain. Or, from what has been said in the preceding Section, a method of arranging surfaces would be obtained by previously arranging systems of lines, since, eliminating the parameter, we should thence deduce the surface wherein they lie.

Such a method, it is evident, will be subject to all the imperfections incident to the arrangement of lines; and on this account will scarcely warrant, as a general method, the praise that it has obtained: under another

aspect, it must be allowed, however, to have considerable claims to our attention. The surfaces afforded by it more nearly resemble those used in the arts, than can be obtained with the same facility from other principles of classification; whilst, in presenting some of the best illustrations of a very difficult branch of the integral calculus, this method of arrangement performs an essential service to an important department of science.

Passing from these general remarks to the theory which gave rise to them, let us assume the three equations

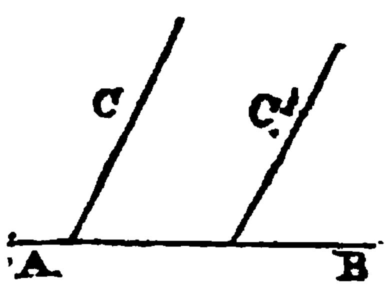
$$\begin{aligned} y &= az + y^0 \\ x &= a'z + x^0 \quad \ldots\ldots 1 \\ y^0 &= a''x^0 + b \end{aligned}$$

where y^0 and x^0, the values of y and x that correspond to $z = 0$, are regarded as parameters, and a, a', a'' as constants, assumed at pleasure, but which remain invariable throughout the reasoning used in relation to the problem.

These equations evidently belong to a system of straight lines, subjected to the restriction imposed by the third equation, and which requires the points wherein the lines intersect the plane of the xy's, to lie in a right line given in that plane.

Such a line C, regarded as moving along a second straight line AB, and constantly preserving a direction parallel to itself, will agree, successively, with every line in the system; and as C by its motion describes a given plane, not only must the lines be restricted to lie in that surface, but the equation of the plane must be deducible from those of the system, by removing the quantities that serve to distinguish the individual lines apart. But the para-

Fig. 255.



meters are the symbols by which this distinction is made, and hence, eliminating the parameters, the result will be an equation common to all the lines; or, it will be the equation of the plane sought.

Now, multiplying the second equation by an indeterminate quantity λ, and adding the three equations together, we obtain

$$y + \lambda x = (a + a'\lambda)\, z + (a'' + \lambda)\, x^{0} + b,$$

where the elimination in question will be effected by assuming $\lambda = - a''$.

Performing the operation, we deduce

$$y - a''x = (a - a'a'')\, z + b,$$

for the equation of a plane expressed in terms of the inclinations of the axes with two lines assumed at pleasure in it, and through one of which the plane of the xy's is made to pass.

267. When a surface is formed in this way, by a line which moves along a second line, the former is called the *generatrix*, and the latter the *directrix*. And, by comparing the preceding example with the examples that occupied our attention in Sec. VI., we observe the connection between the theory of parameters, and the method of classing surfaces by the nature of their generatrixes.

The lines that in the former method are regarded as existing simultaneously, are looked upon in the latter as existing in succession; a single line, the generatrix, replaces, by a change of position, the system before used, and we recognise in the directrix of the present method, the "line of parameters" made use of in the former.

268. In the question solved in art. 266, the directrix was rectilinear, but this restriction is not essential to the reasoning employed, which will apply with equal facility to a directrix of any form.

Assuming this extension of the problem, the equations of art. 266 become,

$$y = az + y^0,$$
$$x = a'z + x^0,$$
$$y^0 = \phi x^0;$$

whence, the first equation may be put under the form

$$\phi x^0 = y - az,$$

or,

$$x^0 = \psi (y - az);$$

and substituting this value of x^0 in the second equation, there results,

$$x - a'z = \psi (y - az) \quad . \; . \; . \; . \; . \; . \; 2$$

as the equation of any surface that can be generated by the motion of a straight line which remains parallel to itself.

Such surfaces are termed cylindric; and may be more conveniently expressed in terms of co-ordinates that are symmetrical, in respect to the known direction of the generating line.

Assuming the axe of the z's to be parallel to this, the ordinate z, art. 254, will disappear from the equation, and as the latter will then contain only two variables, we conclude that every equation of the form

$$y = \mathrm{F}x \quad . \; . \; . \; . \; . \; 3,$$

when taken in reference to the three co-ordinates of space, is the equation of a cylindric surface.

269. Assuming the generating line to pass through a

Now as $\frac{y}{x}$ is the tangent of the angle which the generating line makes with the axe of the x's, if we assume this angle to increase uniformly with the motion of the line in question, along the directrix, or the axe of the z's, we shall have the ordinate z proportional to the angle so described, or

$$z = a \tan.^{-1} \frac{y}{x},$$

the equation of the surface required, and which, by substituting for $\tan.^{-1}$ its value in terms of $\sin.^{-1}$, page 302, assumes the form

$$z = a \sin.^{-1} \sqrt{\frac{y}{x^2 + y^2}}.$$

The equation of a cylinder that has its axis coincident with that of the z's, will be

$$r^2 = x^2 + y^2,$$

and comparing this result with that deduced for the spiral surface, we have

$$r^2 = x^2 + y^2,$$

$$z = a \sin.^{-1} \frac{y}{r}$$

for the equations of their common intersection.

Now the formulæ of page 302 give

$$\sin.^{-1} a = \cos.^{-1} \sqrt{1 - a^2},$$

whence,

$$\sin.^{-1} \frac{y}{r} = \cos.^{-1} \frac{\sqrt{r^2 - y^2}}{r};$$

a result which, from the first of the two equations of the section, may be again reduced to $\cos.^{-1} \frac{x}{r}$; and the

equations of the section may, therefore, be written under the more convenient form

$$z = a \sin.^{-1} \frac{y}{r}$$

$$z = a \cos.^{-1} \frac{x}{r},$$

when we recognize them as the equations of the helix, art. 255.

SECTION VIII.

OF SYSTEMS OF SURFACES.

A single equation between three co-ordinates and a parameter belongs to a simple system of surfaces—equations between three co-ordinates and several parameter, indicate, when the number of the latter exceeds that of the equations, a complex system of surfaces—examples of simple systems.

274. The method of parameters, by which we have been enabled to arrange lines in systems, will apply, with equal facility, to the similar arrangement of surfaces.

Thus, having given an equation

$$z = \phi\,(x, y, p),$$

between three co-ordinates and a parameter, we may eliminate this last, and substitute in its place the value which either of the co-ordinates assumes at the origin.

Choosing for this initial co-ordinate the value of z that corresponds to $x = 0$ and $y = 0$, the elimination in question must be performed between the equations

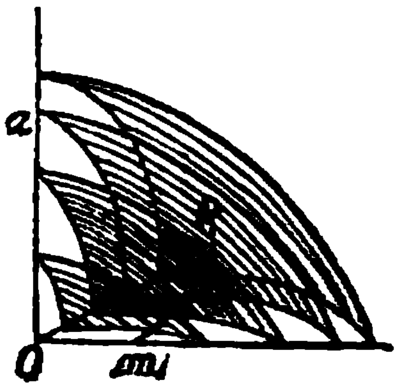
$$z = \phi\,(x, y, p),$$
$$z^{\circ} = \phi\,p\,;$$

and the result will, consequently, assume the form,

$$z = \psi\,(x, y, z^{\circ})$$

which represents a system of surfaces passing into each other by insensible gradations, and distinguished apart by the point wherein they intersect the axe of the z's. Such a system is represented in figure 256, and the position of P, any point in it, is given, "first," by the values of Oa, or z°, which is the parameter of the individual surface wherein the point is situated; and "secondly," by Om, $m\,n$ and nP; or, in other words, by the ordinates x, y and z, that distinguish it from any other point in that surface.

Fig. 256.

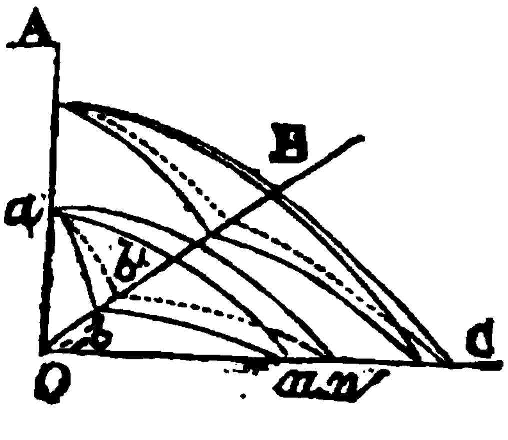


275. Systems of lines, it will be recollected, are classed into simple and complex, and a like division may be made of surfaces.

Simple systems of surfaces are those which have only one parameter, fig. 256, and, consequently, wherein only a definite number of surfaces cut the axe of the parameters in the same point.

Complex systems are those which have more than one parameter, or, in other words, wherein, to each point in an axe of parameters an infinity of surfaces will correspond.

Fig. 257.



Such a system may be represented by the equation

$$F\,(x, y, z, p, p') = 0,$$

or its equivalent equation

$$\phi\,(x,\,y,\,z,\,z^0,\,y^0) = 0,$$

where z^0 has the same meaning as before, and y^0 is the value of y^0 corresponding to $x = 0$ and $z = 0$.

Now taking OA, OB and OC, fig. 257, for the axes of the z's, y's and x's; and assuming $Oa = \gamma$ and $Ob = \beta$, the curved line ab and $b\,n$ may be taken to represent the traces of that surface of the system

$$\phi\,(x,\,y,\,z,\,z^0,\,x^0) = 0,$$

which has the parameters

$$z^0 = \gamma,$$
$$y^0 = \beta.$$

But if, z^0 remaining the same, y^0 becomes Ob', the surface will no longer be that which has occupied our attention, but a new surface of the system, passing, in common with the former, through the point a, but having, in place of $a\,b$ and $b\,n$, the traces $a\,b'$ and $b'n'$.

By merely, therefore, varying y^0, whilst z^0 remains the same, we may arrange together an infinity of surfaces into a "simple" system, every surface of which passes through the point a.

A similar result will follow for any other value of z^0; or, in other words, for every point that can be assumed in the axe of the z's.

And, connecting these remarks, we perceive a complex system to consist of an infinity of simple systems, arranged together by the medium of a parameter.

276. As a first example of the case discussed in article 274, let there be given the equation

$$x^2 + \left(\frac{y}{y^0}\right)^2 + \left(\frac{z}{z^0}\right)^2 = 1 \ldots\ldots 10.$$

This example containing two arbitrary quantities, it may be taken to represent either a simple or a complex system, according as we regard one, or both of these quantities as parameters. Upon the latter hypothesis, the equation would represent every ellipsoid which has one of its axes equal to unity; but as our present object is to illustrate the subject under consideration by an example chosen from among simple systems, we will reduce the problem to that form by assuming one of the parameters as a function of the other.

Let us suppose, for example, that y^0 and z^0 are connected by the equation

$$y^0 = \frac{1}{z^0};$$

the equation 10 will then become

$$x^2z^{02} + y^2z^{04} + z^2 = z^{02}:$$

and making $x = 0$, with the view of obtaining the traces of the surfaces upon the planes of the yz's, we find these traces to constitute a system of curves, represented algebraically, by

$$z^2 = z^{02} - z^{04}y^2;$$

an equation already noticed in art. 258, where the lines included under it were delineated, in fig. 248.

If the assumed relation between y^0 and z^0 had been expressed by

$$y^0 = \mathrm{F}\,z^0,$$

the system of surfaces would have varied with every

form of F, as we shall perceive more distinctly by observing:

First; That whilst y^0 and z^0 are independent, art. 275, the compound system expressed by the equation 10, is composed of an infinity of simple equations, each distinguished by some assigned value of z^0.

Secondly; That assuming a determinate form for F, the equations

$$x^2 + \left(\frac{y}{y^0}\right)^2 + \left(\frac{z}{z^0}\right)^2 = 1,$$

$$y^0 = \mathrm{F}\, z^0,$$

express a simple system not found among the systems obtained by regarding y^0 and z^0 as independent parameters, but formed by taking one surface out of each of these systems.

And, thirdly; That surfaces so chosen will vary with F.

As these remarks are explained at some length in the preceding article, we shall not dwell further upon them; but proceed to illustrate the subject under discussion by a second example.

The problem that we shall select for this purpose, requires us to determine the equation, and examine the relations, of a system composed of all the planes that pass through a given point.

The first of these objects is easily effected, since, taking the given point as the origin, the equation of the system, art. 224, will be

$$z = ax + by,$$

where a denotes the tangent of the angle which the trace on the plane of the xz's makes with the axe of the x's;

and where b denotes the tangent of the corresponding angle on the plane of the yz's.

Taken by itself, this equation denotes a complex system, and as our present object is to obtain a system that is simple, we must limit the parameters, by assuming them to be mutually functions of each other; this relation, however, is perfectly arbitrary, and the equations

$$z = ax + by,$$
$$b = \mathrm{F}\,a,$$

may, therefore, be taken to represent any system of planes that has the property required.

To illustrate the subject further, let us assign to the function F a particular form, the form, for example, found to exist between the co-ordinates of a circle; this done, and putting for a and b the letters p and p', we have, in place of the preceding equations,

$$z = px + p'y,$$
$$r^2 = p^2 + p'^2,$$

which express a system readily understood from some of its characteristic properties.

Thus, denoting a perpendicular to any plane of the system by the letter r, and comparing the first of the two equations with the equation of a plane, given in art. 224, we find

$$p = \frac{\cos.\, rx}{\cos.\, rz}$$

$$p' = \frac{\cos.\, ry}{\cos.\, rz};$$

and, squaring and adding the results, there arises,

$$r^2 = \frac{\cos.\,^2 rx + \cos.\,^2 ry}{\cos.\,^2 rz}$$

which, having regard to the equation

$$\cos.^2 rx + \cos.^2 ry + \cos.^2 rz = 1,$$

reduces to,

$$1 + r^2 = \frac{1}{\cos. rz^2},$$

or,

$$r^2 = \tan.^2 rz\,;$$

whence it appears that all the planes in the system are inclined at a constant angle to the axe of the z's.

CHAPTER II.

RELATIONS THAT EXIST BETWEEN THE LINES OR THE SURFACES OF ONE SYSTEM, AND THOSE OF ANOTHER.

PRELIMINARY REFLECTIONS.

We discovered, in discussing the properties of the circle and the sphere, certain lines and planes named tangents, and tangent planes, that had remarkable relations with those figures.

INQUIRIES SUGGESTED BY THESE REFLECTIONS.

Do all curves and curve surfaces admit of tangents and tangent planes?

SECTION I.

OF TANGENTS AND NORMALS OF PLANE CURVES.

Definition of a tangent—equations of the tangents of a plane curve—examples—polar equations of tangents—examples—asymptotes—examples—of normals—examples—of lines making known angles with curves—examples.

277. In discussing the properties of the circle, art. 162, we sought its relations with a system of straight lines diverging from a given point; and finding these lines, when the point is placed without the circumference, to divide themselves into two classes, according as they intersect the circle, or pass beyond that curve, we came to the knowledge of a line not arranged under either of the classes, and remarkable as a limit by which the latter are divided.

This line was named the tangent.

And as, on further examining the properties of such lines, and the relation they bear to the circle, we find them to assign, at every point of the curve, its deflection from a rectilinear course, it becomes an object worthy of consideration whether all curves are not susceptible of a similar relation.

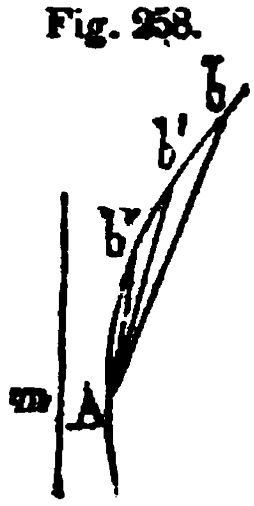
Chap. II. Relations that exist between the lines or the surfaces of one system and those of another.

Art. 277. Definition of a tangent.

The nature of the latter, an inquiry into which we shall be called upon to enter, will be found to involve a metaphysical difficulty; the definition of a tangent assumes it to be a right line that agrees in direction with a curve, and the mind cannot readily perceive an agreement of direction between a curve which continually deflects from its course, and a straight line, the direction of which is every where the same. To place this subject in its true light, it will be necessary for us to state, with some detail, the view of it that is commonly taken.

Having assigned the point of tangency, A, it is usual to assume points b, b', b'', &c. in the curve; and to argue, that as the points approach towards A, the directions of the chords will approach to that of the curve itself, since the chords, becoming at last of insensible magnitude, may be ultimately considered as blending themselves with the former; a case, however, that only happens when b and A become identical, or when the chord ceases to exist.

Fig. 258.



The difficulty met with in this reasoning depends upon the idea denoted by the term "direction." Regarding this idea as implying the direction of a line uniting two points, the direction of the chord Ab will be intelligible only so long as the chord exists, and will become altogether without meaning, when the point b is assumed to coincide with the point of tangency.

The direction m, to which the chords Ab, Ab', &c. continually tend as b approaches A, is not, therefore, to be regarded as a position ever attained by the latter, but merely as a limit whereto the directions of the

chords approach to within less than any assignable difference.

From this reasoning it will follow, that however small the portion of the curve examined, a perfect agreement between it and the line is not, as the vague expressions of many writers would seem to imply, included in the idea of a tangent; the latter being merely a line limiting the directions of the chords drawn through a given point in the curve.

Seen under this view, the imperfect relation between a straight line and a curve will appear sufficiently intelligible; but a phraseology may be employed that is still less objectionable, and which seems to remove all difficulty, since we may define a tangent as a line which agrees with a curve, at a given point of the latter, "more nearly" than any other straight line.

Adopting this definition, we shall not only perceive that every curve admits the relation comprehended under the idea of a tangent, but shall be enabled without difficulty to find the position of the latter.

The differential calculus, which is founded on the notion of omitting those parts of quantities that can be made indefinitely less than the parts retained, is excellently adapted to express the imperfect relations now under discussion, and has accordingly been used for that purpose from the period of its first invention.

The process is as follows.

278. Putting the equation of the curve under the general form,

$$y = \mathrm{F}x$$

and expressing by

Chap. II. Relations that exist between the lines or the surfaces of one system, and those of another.

Art. 278. Equations of the tangents of a plane curve.

$$y = ax + b$$

the equation of any straight line; the condition restricting the latter to pass through the point of tangency, will require the equation of the line to be fulfilled by the co-ordinates belonging to the point.

Denoting these co-ordinates by x' and y', we have the simultaneous equations

$$\begin{aligned} y &= ax + b \\ y' &= ax' + b' \\ y' &= \mathrm{F}x', \end{aligned}$$

where the two latter merely assert the point of tangency to be common to the curve and line.

Now subtracting the second of the equations from the first, there results

$$y - y' = a\,(x - x') \quad . \; . \; . \; . \; \alpha$$

as the equation of all right lines passing through the point in question.

The second condition which requires the curve to agree with the line more nearly than with any other straight line, will be most conveniently investigated by examining the ordinates in the immediate neighbourhood of the point of tangency. By such a research we shall discover the departure of the line from the curve, and, rendering this the least possible, shall arrive at the parameter, (equation α) belonging to the line which has the property in question.

Now assuming the abscissa x' of the point of tangency to increase by the indeterminate quantity h, and representing, respectively, by $y_{,}$ and y, those ordinates of the curve and line that correspond to the abscissa so increased; we shall have by the theorem of Taylor,

$$y_{,} = y' + \frac{dy'}{dx'}\frac{h}{1 \cdot 2} + \frac{d^2y'}{dx'^2}\frac{h^2}{1 \cdot 2} + \&c.$$

$$y = y' + \frac{dy}{dx}h,$$

and subtracting the latter from the former, the departure of the curve from the line will be expressed by the difference

$$y_{,} - y = \left(\frac{dy'}{dx} - \frac{dy}{dx}\right)h + \frac{d^2y'}{dx'^2}\frac{h^2}{1 \cdot 2} + \&c.$$

and as the equation α places at our disposal only one parameter a, we must render the difference here found, the least possible, by so assuming a as to destroy the greatest term which is found in the value of

$$y_{,} - y.$$

But it is shown by the writers on the differential calculus, that in any series which proceeds by the ascending powers of an indeterminate quantity h, the value of h can be taken so small as to render the first term of the series greater than the sum of all the remaining terms; and it will therefore follow that $y' - y$ becomes the least possible when a is so assumed as to make the term

$$\left(\frac{dy'}{dx'} - \frac{dy}{dx}\right)h$$

to disappear from its value.

This result will be obtained by assuming

$$\frac{dy'}{dx'} = \frac{dy}{dx};$$

and we have only to show that a can be taken in such a manner as to agree with this assumption.

But differentiating the equation α, there arises

$$a = \frac{dy}{dx},$$

and making this substitution, and bearing in mind that we have taken the differentials at the point of tangency, where x and y become x' and y', we obtain

$$y - y' = \frac{dy'}{dx'}(x - x') \ . \ . \ . \beta$$

for the equation of the tangent sought.

279. Let it be required, for example, to determine the equations of the systems of lines which are tangential to the circle.

Assuming the origin at the centre, the equation of the curve will be

$$x'^2 + y'^2 = r^2$$

whence, by differentiation,

$$x'dx' + y'dy' = 0$$

$$\text{or } \frac{dy'}{dx'} = -\frac{x'}{y'};$$

substituting which in the equation β, we have

$$y - y' = -\frac{x'}{y'}(x - x')$$

or

$$yy' + xx' = r^2$$

for the equation of any line that is tangential to the circle.

The parameters y' and x' are quantities that remain the same while the point of tangency is invariable; and in this respect differ from x and y, which are the coordinates whereby the several points in the tangents are distinguished.

Sect. I. Of tangents and normals of plane curves.
Art. 279. Examples.

As the equation of the circle, whilst the origin is taken at the centre, does not vary with the position of the axes, we may assume the axe of the x's to pass through the point of tangency; a condition that will give the values

$$x = r', \quad y' = 0,$$

and consequently reduce the equation of the tangent to

$$x = r,$$

a result that indicates (art. 216) a line parallel to the axe of the y's, or perpendicular to the radius which passes through the point of tangency (art. 162). As a second example, let us assume the curve to be expressed by the equation

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1;$$

whence

$$\frac{dy'}{dx'} = -\frac{b^2}{a^2} \cdot \frac{x'}{y'}$$

which gives, for the equation of the tangent,

$$y - y' = -\frac{b^2}{a^2} \cdot \frac{x'}{y'} (x - x')$$

or,

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1.$$

The points wherein the tangents meet the axes will be found by assuming x or y equal to zero; adopting the latter assumption, and substituting this value in the equation last found, we have

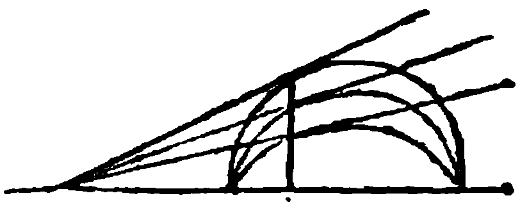
$$x = \frac{a^2}{x'}$$

for the ordinate of the point wherein the tangent meets the axe of the x's.

And as this result is independent of b, we deduce a

remarkable relation of the tangents belonging to the system of curves expressed by the given equation. Observing with this view, that when a and b are real and positive, the equation in question is that of an ellipse, and bearing in mind what has been said above concerning the intersection of the tangent and the axe, we conclude that in a system of concentric ellipses having the same major axe, the tangents corresponding to a given abscissa x', would all intersect the axe of the x's in the same point. This proposition would also hold for a system of hyperbolas, or for the more general case wherein the system was composed of any curves of the second degree, provided, however, they were concentric, and had the same major axis.

Fig 259.



280. The analysis used to determine the lines tangential to a curve, will be equally applicable when the equation of the latter is given in terms of other co-ordinates.

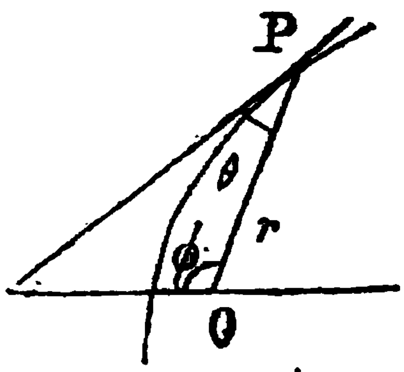
As, for example, when the equation is polar, or is expressed in terms of a radius vector r', and the angle ϕ, which the latter makes with some primordial line.

The equation to the curve may then be written under the form

$$r' = F\,\phi',$$

and the equation to the tangent under a similar form: whence, expanding by Taylor's Theorem, and reasoning, as before, we shall deduce, in the case of tangency,

Fig. 260.



$$\frac{dr'}{d\phi'} = \frac{dr}{d\phi}$$

where the ordinates r' and ϕ' belong to the curve, and the ordinates r and ϕ to the line.

Now if θ is taken to denote the inclination of the tangent to the radius vector, the three co-ordinates r, ϕ and θ, will suffice to assign the position of the tangent; and as the two first of these quantities will be given whenever the point of tangency is known, it is only necessary, in order to complete the analysis of the problem, to assign θ by means of the equation $\frac{dr'}{d\phi'} = \frac{dr}{d\phi}$.

But whilst the line remains invariable, we have

$$\theta + \phi = c$$

$$r \sin.\ \theta = c'$$

where c and c' are constants.

From the first of these equations there is obtained

$$d\,\theta = -\,d\,\phi,$$

and from the second

$$dr \sin.\ \theta = -\,r \cos.\ \theta\ d\,\theta,$$

or,

$$\frac{dr}{d\theta} = -\,\frac{dr}{d\phi} = -\,r \cot.\ \theta;$$

whence,

$$\cot.\ \theta = \frac{1}{r'}\frac{d\,r'}{d\phi'};$$

and the position of the tangent is completely assigned.

281. Let it be required, for example, to determine in this manner the tangents of a circle.

The equation of the latter will be

$$r = c;$$

whence,

$$\frac{dr'}{d\phi'} = \frac{dr}{d\phi} = 0$$

and

$$\cot.\ \theta = \frac{1}{r'} \cdot \frac{dr'}{d\phi'} = 0,$$

a result that gives

$$\theta = \tfrac{1}{2}\pi.$$

As a second example of the same kind, let it be required to determine the tangents of an ellipse.

The polar equation of the curve, art. 241, is

$$r' = \frac{m}{1 + e \cos.\ \phi'}$$

and if we denote by r'' the radius drawn to the other focus, and by ϕ'', the angle which this radius makes with the major axe, we shall have, when the angles ϕ' and ϕ'' are taken towards the same side,

$$r'' = \frac{m}{1 - e \cos.\ \phi''}.$$

Inverting these formulæ, and differentiating, we obtain

$$\frac{1}{r'}\frac{dr'}{d\phi'} = e\, r' \sin.\ \phi'$$

$$-\frac{1}{r'}\frac{dr''}{d\phi''} = e\, r'' \sin.\ \phi''.$$

But

$$r' \sin.\ \phi' = r'' \sin.\ \phi'',$$

whence,

$$\frac{1}{r'}\frac{dr'}{d\phi'} = -\frac{1}{r''}\frac{dr''}{d\phi''}$$

and consequently,

$$\cot.\ \varphi' = -\cot.\ \varphi''$$

or,

$$\theta' = \tfrac{1}{2}\pi - \theta''$$

The angles θ' and θ'' are, respectively, the inclinations of the tangent with r' and with r''; and, as in the preceding analysis these angles are measured towards the same side, if we measure them towards opposite sides, the last equation will become

$$\theta' = \theta'',$$

whence the tangent is seen to make equal angles with the radii drawn from the two foci.

282. As the position of the tangent at any point, determines the direction of the curve, the theory of tangents is an essential branch of the analysis of curve lines, and it will be necessary, as we proceed with the present inquiry, to examine the most important applications of this theory that present themselves: a particular case, in which the point of tangency is assumed at an infinite distance from the origin, is especially useful in determining the infinite branches of a curve; and will be conveniently discussed before we proceed further.

The object of the inquiry is to determine whether the infinite branches of a curve retain their character at great distances from the origin, or tend, as in the case of the hyperbola, to approach towards right lines.

Upon the latter hypothesis, a branch of the curve, and the tangents of these distant points in it, would approach towards the same straight line as their limit; and the hypothesis will therefore be tested by examining, from the equations of the tangents, whether the latter do in-

tangency recedes from it, and being also, as we have shown, parallel to a given direction, become at last indefinitely near to a line, which they yet never attain: this line is an asymptote to the curve, and its position will be determined by the condition that it passes through the origin, and makes with the axe of the x's the angle $\tan.^{-1}\frac{b}{a}$.

If we take as a second example the cissoid of Diocles, a curve expressed by the equation

$$y'^2 = \frac{x'^3}{a - x'},$$

we shall have

$$y'^{-2} = ax'^{-3} - x'^{-2};$$

whence,

$$\frac{dy'}{dx'} = \frac{3a - 2x'}{2x'^4}\,y'^3.$$

But, from the equation of the curve, y' is infinite when $x' = a$, and with these values of x' and y', we obtain

$$\frac{dy'}{dx'} = \infty.$$

Making the same substitution in the equation

$$x = x' - y'\,\frac{dx'}{dy'}$$

there results

$$x = a,$$

whence a line parallel to the axe of the y's, and at the distance a, is an asymptote to the curve.

284. The theory of lines that make a given angle

with a curve, will follow, by an easy transition from the theory of lines in which that angle is zero; but previously to entering upon the general inquiry, we shall examine the particular case wherein the angle in question is a right angle.

Such lines are called *normals*, and are of frequent use in analysis.

The method of determining their position will be merely an application of the proposition demonstrated in art. 219. Thus, assuming x' and y' as the co-ordinates of the point wherein the curve and line meet, the equation of any straight line passing through that point will be

$$y - y' = a(x - x')$$

where a is a parameter distinguishing the individual lines in the group. And assuming any two of these lines to be mutually perpendicular, they can be expressed, as we learn from the article in question, by means of the equations

$$y - y' = a(x - x')$$

and

$$y - y' = -\frac{1}{a}(x - x').$$

Now, by taking a equal to $\frac{dy'}{dx'}$ the first of these lines will become a tangent, and as the normal is a line perpendicular to this last, and passing through the point of tangency, we have

$$y - y' = -\frac{dx}{dy}(x - x')$$

for the equation of the normal.

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Art. 285. Examples. Art. 286. Polar equation of normals.

285. Let it be required for example, to determine the normal to any point of a circle.

The equation of the latter is

$$y'^2 = r^2 - x'^2,$$

whence,

$$\frac{dy'}{dx'} = -\frac{x'}{y'},$$

and substituting in the equation of the normal, this last becomes

$$y - y' = \frac{y'}{x'}(x - x');$$

or,

$$yx' - y'x = 0.$$

As $y = 0$ and $x = 0$ satisfy this equation, the line denoted by it passes through the centre, and the normal is therefore a radius.

286. The equation of the normal, as that of the tangent, may be expressed in terms of a polar system of co-ordinates. In fact, as the tangent and normal are mutually rectangular, the inclination of the first to the radius vector will be the complement of the inclination of the second to that line. Whence, denoting the last of these angles by θ, we have, art. 280,

$$\tan.\ \theta = \frac{1}{r'}\frac{dr'}{d\phi'}.$$

287. Applying this formula to determine the normal at any point of an ellipse, we have for the equation of the latter, art. 241,

$$r' = \frac{m}{1 + e \cos. \phi'};$$

whence,

$$\frac{1}{r'} \frac{dr'}{d\phi'} = e\, r' \sin. \phi' = \tan. \theta;$$

and substituting for r' its value, we obtain

$$\tan. \theta = \frac{e\ m\ \sin. \phi'}{1 + e \cos. \phi'};$$

which determines the normal.

It is often, however, convenient to express the position of the normal of the ellipse, in terms of a radius vector, drawn from the centre of the curve. Denoting this radius by r, and its inclination to the major axe by ϕ, we have

$$r^2 = \frac{a^2 - a^2}{1 - e^2 \cos. \phi^2},$$

whence,

$$\frac{1}{r} \frac{dr}{d\phi} = \tan. \theta = -\frac{e^2 \cos. \phi \sin. \phi}{1 - e^2 \cos. ^2\psi}.$$

Supposing the ellipse to revolve about the minor axis, the solid generated would be an oblate spheroid ; and as, by observing a due proportion between the axes, this solid could be made to coincide with the figures of the earth, the proposition we are illustrating would enable us to determine the direction of the "plumb line," or of a line perpendicular to the earth's surface. The value of e, in this case, is extremely small, and the preceding equation may, therefore, without sensible error, be written

$$\tan. \theta = -\tfrac{1}{2}\, e^2 \cos. \phi \sin. \phi,$$

or,

$$\tan.\ \theta = -\tfrac{1}{2}\ e^2 \sin.\ 2\ \phi\ ;$$

and as it appears from this result that θ will also be very small, we may put the equation under the form

$$\sin.\ \theta = -\tfrac{1}{2}\ e^2 \sin.\ 2\ \psi:$$

whence there arises for the distance δ, of the normal, from the centre of the ellipsoid,

$$\delta = r \sin.\ \theta = -\frac{e^2}{2} \sin.\ 2\,\mathrm{L}\ ;$$

where L is the latitude, or the distance from the equator to that point on the surface through which the normal is drawn.

288. Resuming the more general case, wherein the line sought makes any given angle α with the curve; let us denote by m and m' the tangent and the line which is sought: the expression, art. 100,

$$\cos.\ mm' = \cos.\ mx \cos.\ m'x + \sin.\ mx \sin.\ m'x,$$

will then become

$$\cos.\ \alpha = \cos.\ (m'x - mx)\ ;$$

whence,

$$m'x = \alpha + mx\ ;$$

a result sufficiently apparent from other considerations.

Now the equation of a right line m', that passes through a point in the curve, will be

$$y - y' = \tan.\ m'x\ (x - x'),$$

where y' and x' are the co-ordinates of the point in question.

Hence, substituting for tan. $m'x$ its value, tan. $(\alpha + mx)$,

replacing tan. mx by $\frac{dy}{dx}$, art. 278, and denoting tan. α by a, we obtain,

$$y - y' = \frac{dy' - adx'}{dx' - ady'}(x - x'),$$

for the equation of the line sought.

But the position of this line may be more readily determined by previously calculating that of either the tangent or the normal: one or other of these lines may then be regarded as a new axe, and taking the point of intersection with the curve as a new origin, the line sought will pass through this origin, and make a known angle with the axe so obtained. Or, if it is convenient to express the position of the line sought without the intersection of the tangent or normal, we may either employ the expression already deduced for that purpose, or transform it into another which contains only polar co-ordinates.

Employing, with this view, ϖ and θ to represent the angles which the radius vector makes with the line sought, and with the normal, and putting α for the angle intercepted between these two last lines, we have

$$\varpi = \theta - \alpha,$$

and

$$\tan.\,\varpi = \frac{\tan.\,\theta - \tan.\,\alpha}{1 + \tan.\,\theta\,\tan.\,\alpha}:$$

a result that may be written, art. 286, under the form

$$\tan.\,\varpi = \frac{dr' - ar'd\phi}{r'd\phi + a\,dr'}.$$

289. Let the curve, for example, be the common parabola; and let it be required to determine the posi-

tion of a line which shall have the same inclination to the curve, as the radius vector drawn from the focus.

The equation of the curve, expressed in terms of the radius vector in question, is, art. 241,

$$r' = \frac{2\beta}{1 + \cos.\,\phi},$$

and inverting and differentiating,

$$\frac{1}{r'}\frac{d\,r'}{d\,\phi} = \frac{1}{2\,\beta}\sin.\,\phi,$$

or,

$$\tan.\,\theta = \frac{\sin.\,\phi}{1 + \cos.\,\phi} = \tan.\,\tfrac{1}{2}\,\phi,$$

or,

$$\theta = \tfrac{1}{2}\,\phi.$$

But in the case before us, the angle $-\theta$ is that intercepted between the line sought and the normal; whence, substituting in the expression for tan. ϖ, $-\theta$ in place of α, the expression becomes

$$\tan.\,\varpi = \frac{2\tan.\,\theta}{1 + \tan.\,\theta^2} = \tan.\,2\,\theta,$$

and replacing θ by the value above obtained, there results,

$$\tan.\,\varpi = \tan.\,\phi,$$

or,

$$\varpi = \phi,$$

and the line sought is parallel to the axis of the curve.

SECTION II.

OF TANGENTS AND OF NORMAL PLANES TO LINES SITUATED IN SPACE.

Equations of the tangents of lines given in space—examples—of normal planes.

290. The method used to determine the tangents of lines situated in a given plane, will extend to the general case of the problem, and enable us to determine the tangents of every species of line, whether of single or of double curvature, and without regard to the parts of space in which they may be situated.

The truth of this assertion will become apparent, by referring to the process used in art. 278, and adapting it to the problem now under consideration.

Thus, supposing the curve to be represented by the equations

$$z' = \mathrm{F}x'$$
$$y' = fx';$$

and taking

$$z = ax + b,$$
$$y = a'x + b',$$

to be the equations of the straight line sought, we shall have, as in art. 278,

$$z_{\prime}' = z' + \frac{dz'}{dx} \cdot h + \frac{d^2z'}{dx^2} \cdot \frac{h^2}{1 \cdot 2} + \&c.$$

$$y_{\prime}' = y' + \frac{dy'}{dx} \cdot h + \frac{d^2y}{dx} \cdot \frac{h^2}{1 \cdot 2} + \&c.$$

where h is an arbitrary increase assigned to x, and $z_{\prime}'$ and $y_{\prime}'$ are the values of z' and y' corresponding to the abscissa $(x' + h)$.

Denoting, in like manner, by $z_{\prime}$ and $y_{\prime}$ the values of z and y that correspond to $x + h$, we have,

$$z_{\prime} = z + \frac{dz}{dx} \cdot h,$$

$$y_{\prime} = y + \frac{dy}{dx} \cdot h.$$

Whence, since $z = z'$, and $y = y'$,

$$z_{\prime}' - z_{\prime} = \left(\frac{dz'}{dx'} - \frac{dz}{dx}\right) \cdot h + \frac{d^2z'}{dx'^2} \frac{h^2}{1 \cdot 2} + \&c.$$

$$y_{\prime}' - y_{\prime} = \left(\frac{dy'}{dx'} - \frac{dy}{dx}\right) \cdot h = \frac{d^2y'^2}{dx^2} \frac{h^2}{1 \cdot 2} + \&c.$$

and reasoning as in art. 278, the difference between the ordinates of the line and the curve will be the least possible, when the terms containing the first power of h disappear; or when we have the equations

$$\left.\begin{aligned} \frac{dz'}{dx'} - \frac{dz}{dx} &= 0, \\ \frac{dy'}{dx'} - \frac{dy}{dx} &= 0, \end{aligned}\right\} \quad \ldots\ldots\ \alpha.$$

Now the condition that requires the straight line to pass through the point assigned, by the co-ordinates x', y' and z', reduces the equations of the line, art. 278, to the form

$$z - z' = a(x - x')$$
$$y - y' = a'(x - x');$$

where, differentiating,

$$a = \frac{dz}{dx},$$

$$a' = \frac{dy}{dx}.$$

And substituting these values, and having regard to the equations α, there arises,

$$\left.\begin{aligned} z - z' &= \frac{dz'}{dx'}(x - x') \\ y - y' &= \frac{dy'}{dx'}(x - x') \end{aligned}\right\} \quad \ldots. \beta$$

for the equations of the tangent sought.

The principle used in this investigation is the same as that employed for plane curves: the straight line and the curve are assumed to have a point in common, and their departure at any distance from this point is estimated by the difference between the co-ordinates, and which difference is finally made the least possible. It will be observed, however, that in estimating the departure, we have measured it in the directions of three known axes, x, y and z, without inquiring whether the minima departures, in these directions, insure the least departure in every other; but, that such is the fact, may be shown by the formulæ for transforming co-ordinates, art. 101, where, it will immediately be seen, that

when the difference of the ordinates of the line and curve is of the order h^2, for each of the axes x, y and z, it will be of the same order for every other ordinate: and the agreement between the line and curve, will thus be the nearest possible.

291. Let the given curve, for example, be the helix, art. 255.

The equations are

$$z' = m \cos.^{-1} \frac{x'}{r}$$

$$z' = m \sin.^{-1} \frac{y'}{r}$$

whence

$$\frac{dz'}{dx'} = -\frac{m}{y'}, \quad \frac{dz'}{dy'} = \frac{m}{x'};$$

and the equations β, of the tangent, become

$$z - z' = -\frac{m}{y'}(x - x')$$

$$y - y' = \frac{x'}{y'}(x - x')$$

The tangent of the inclination of the curve to the plane of the xy's is evidently expressed by the equation

$$\tan. i = \sqrt{\left(\frac{dz'^2}{dx'^2 + dy'^2}\right)}$$

and substituting in this expression the values of $\frac{dz'}{dx'}$ and $\frac{dz'}{dy'}$, we obtain

$$\tan. i = \frac{m}{r}$$

whence it appears that the inclination of the curve to the plane of the xy's is constant.

292. The normal of a curve given in space may also be investigated upon the same principles as the normal of a curve restricted to lie in a plane, but the result indicates that an infinity of normals can be drawn to a curve through any point of the latter. The problem, in fact, is the same as that discussed in art. 218; and if in the equation

$$mm' + nn' + 1 = 0$$

there deduced, we substitute for m and n their values, $\frac{dx'}{dz'}$ and $\frac{dy'}{dz'}$, obtained from the equation to the tangent, the result will be the only condition whereby m' and n' are restricted; and as there will then be two indeterminate quantities m' and n', and but one equation to be fulfilled, we conclude, as in the article alluded to, that innumerable values of these co-efficients will satisfy the given conditions; or, as is also concluded in the same article, that if a plane is drawn at right angles to the curve, the latter will be a normal to every straight line that lies in the plane.

The methods of determining the asymptotes of curves given in space, and the lines that form known angles with them, follow so readily from the theory given in sect. 1, that it will be unnecessary to enter further upon the subject.

SECTION III.

OF THE TANGENT PLANES AND NORMALS OF SURFACES.

Equation of the tangent plane to a surface—examples—equations of the normal to a surface—examples—inclination of the tangent planes to either of the co-ordinate planes, and of the normal to either of the axes.

293. The definition that has been given of the line that is tangential to a curve, is equally applicable to a surface and a plane. The tangent planes of the sphere have already occupied our attention, and it now only remains to generalize the problem, and seek among all the planes that meet a surface in a given point, that which has the nearest coincidence with the surface.

Representing the surface and the plane by the equations

$$z' = \phi\,(x', y')$$

and

$$z = ax + by + c$$

the value of z' that corresponds to the co-ordinates $x' + h$ and $y' + k$ will be

$$z_{\prime}' = z' + \frac{dz'}{dx'} h + \frac{dz'}{dy'} k + \&c.$$

and the value of z that corresponds to the same co-ordinates is

$$z_{\prime} = z' + \frac{dz}{dx} h + \frac{dz}{dy} k;$$

whence, by subtraction,

$$z_{\prime}' - z_{\prime} = \left(\frac{dz'}{dx'} - \frac{dz}{dx}\right) h + \left(\frac{dz'}{dy'} - \frac{dz}{dy}\right) k + \&c.$$

And as the terms which follow the two first, contain higher powers of h and k, the values of the latter may be taken sufficiently small for these terms to exceed all the remainder of the series.

Whence, limiting ourselves to very small values of h and k, it is manifest that we shall make $z_{\prime}' - z_{\prime}$ the least, by destroying as far as possible, these two first terms of the series.

Now the equation of the plane contains three arbitrary constants, a, b and c; one of which, c, is determined by the condition that requires the plane to pass through a given point in the surface, or to make $x = x'$, $y = y'$ and $z = z'$; but substituting these values, the equation of the plane becomes

$$z' = ax' + by' + c,$$

and subtracting this result from the expression

$$z = ax + by + c$$

there arises

$$z - z' = a(x - x') + (y - y') \; . \; . \; . \; . \; . \; . \; 1,$$

3 z

for the equation of the plane that passes through the point required.

And as this expression still contains two arbitrary constants, the condition which requires the first terms to disappear from the difference $z_{,}' - z$ can be fulfilled. In fact, differentiating the equation of the plane, we have

$$\frac{dz}{dx} = a, \quad \frac{dz}{dy} = b,$$

whence, substituting for $\frac{dz}{dx}$ and $\frac{dz}{dy}$ their values $\frac{dz'}{dx'}$ and $\frac{dz'}{dy'}$, obtained from the condition in question, the expression 1 will become

$$z - z' = \frac{dz'}{dx'}(x - x') + \frac{dz'}{dy'}(y - y') \; . \; . \; . \; . \; . \; 2,$$

which representing a plane that fulfils all the conditions required by the definition of the tangent plane, is the equation of the latter.

294. Let it be required, for example, to determine the equation of the plane that is tangential to the common cone at a given point on its surface.

The equation of the cone, when the origin is taken at the vertex, and the axe of the z's coincides with the axis, is, art. 246,

$$x'^2 + y'^2 = a^2 z'^2;$$

whence

$$\frac{dz'}{dx'} = \frac{x'}{a^2 z'}, \quad \frac{dz'}{dy'} = \frac{y'}{a^2 z'},$$

and the equation 2′ of the tangent plane, becomes

$$z - z' = \frac{x'}{a^2 z'}(x - x') + \frac{y'}{a^2 z'}(y - y').$$

Clearing this expression of fractions, and having regard to the equation of the surface, we obtain

$$a^2 zz' = xx' + yy',$$

and as this is fulfilled when x, y and z are each made equal to zero, the tangent planes of all the points in the surface pass through the origin; and must, therefore, touch the surface in the straight line determined by the latter, and the point of tangency.

Taking, as a second example, the ellipsoid, art. 249, we have the equation

$$\left(\frac{x'}{k'}\right)^2 + \left(\frac{y'}{k''}\right)^2 + \left(\frac{z'}{k'''}\right)^2 = 1;$$

whence,

$$\frac{dz'}{dx'} = -\left(\frac{k'''}{k'}\right)^2 \cdot \frac{x'}{z'},$$

$$\frac{dz'}{dy'} = -\left(\frac{k'''}{k''}\right)^2 \cdot \frac{y'}{z'};$$

and the equation of the tangent plane becomes

$$z - z' = -\left(\frac{k'''}{k'}\right)^2 \cdot \frac{x'}{z'}(x - x' - \left(\frac{k'''}{k''}\right)^2 \cdot \frac{y'}{z'}(y - y')$$

or, clearing of fractions and simplifying,

$$\frac{xx'}{k'^2} + \frac{yy'}{k''^2} + \frac{zz'}{k'''^2} = 1.$$

295. The equations of the normal to a surface, or of the line which is perpendicular to the tangent plane, may be found from art. 232, where the projections of the

perpendicular are shown to be at right angles to the traces of the plane.

Assuming

$$ax + by + cz + d = 0,$$

as the equation of the latter, the traces on the planes of the zx's and zy's will be represented by the equations

$$z = \frac{a}{c}x - \frac{d}{c}$$

$$z = \frac{b}{c}y - \frac{d}{c};$$

or,

$$z = \frac{dz}{dx}x - d'$$

$$z = \frac{dz}{dy}y - d.$$

But from art. 219, the equations of lines perpendicular to these may be derived from them, by inverting the coefficients of x and y and changing their signs; and since in the case before us we have $\frac{dz}{dx} = \frac{dz'}{dx'}$ and $\frac{dz}{dy} = \frac{dz'}{dy'}$, the equations of the normal, or of a line perpendicular to the tangent plane, and passing through the point of contact, will be,

$$\left.\begin{aligned} x - x' &= -\frac{dz'}{dx'}(z - z') \\ y - y' &= -\frac{dz'}{dy'}(z - z'). \end{aligned}\right\} \quad \ldots\ldots 3$$

296. Let us take, to illustrate this subject, the same examples that were used in art. 294.

From the first we have,

$$\frac{dz'}{dx'} = \frac{1}{a^2} \cdot \frac{x'}{z'}, \quad \frac{dz'}{dy'} = \frac{1}{a^2}\frac{y'}{z'};$$

and substituting in the equations of the normal, and simplifying, they become,

$$x'z + a^2xz' = x'z' + a^2x'z',$$
$$y'z + a^2yz' = y'z' + a^2y'z'.$$

If we assume, in the first of these equations, that x is zero, and substitute the result so obtained in the second, it will be found that y is also zero; whence, the normal to any point of the surface passes through the axis.

297. The inclinations of the tangent plane to the several co-ordinate planes, and of the normal to the several axes—elements of frequent use in analysis, may be readily obtained from what has gone before.

These elements, we may remark, are not distinct; since the angle formed between the tangent plane and either of the co-ordinate planes, is measured by the inclination of their perpendiculars; or, in other words, by the inclination of the normal to one of the axes. Now, to determine the latter, denote the normal by n, and put the equation, art. 214,

$$\cos. nx^2 + \cos. ny^2 + \cos. nz^2 = 1,$$

under the form

$$\left(\frac{\cos. nx}{\cos. nz}\right)^2 + \left(\frac{\cos. ny}{\cos. nz}\right)^2 + 1 = \left(\frac{1}{\cos. nz}\right)^2 \ldots 4.$$

Differentiating the equations of art. 232, and recollecting that n, in the case before us, takes the place of x' in the article alluded to, we have,

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$$\frac{\cos. nx}{\cos. nz} = \frac{1}{\frac{dz}{dx}}, \quad \frac{\cos. ny}{\cos. nz} = \frac{1}{\frac{dz}{dy}} \quad . \; . \; . \; 5,$$

and, from the relation of the surface to the normal

$$\frac{1}{\frac{dz}{dx}} = -\frac{dz'}{dx'}, \quad \frac{1}{\frac{dz}{dy}} = -\frac{dz'}{dy'} \quad . \; . \; . \; . \; 6;$$

whence, substituting these values in the equation 4, there arises,

$$\left(\frac{dz'}{dx'}\right)^2 + \left(\frac{dz'}{dy'}\right)^2 + 1 = \left(\frac{1}{\cos. nz}\right)$$

whence we finally obtain

$$\cos. nz = \frac{1}{\sqrt{\left(\frac{dz'}{dx'}\right)^2 + \left(\frac{dz'}{dy'}\right)^2 + 1}} \quad . \; . \; 7$$

for the cosine of the angle included between the normal and the axe of the z's, an angle we have remarked that is equal to the inclination of the tangent plane to the plane of the xy's.

The angles formed between the normal and the remaining axes, and, consequently, between the tangent plane and the remaining co-ordinate planes, may be obtained from combining the preceding value of cos. nz with the equations 5 and 6, and are,

$$\cos. ny = \frac{-\frac{dz'}{dy'}}{\sqrt{\left(\frac{dz'}{dx'}\right)^2 + \left(\frac{dz'}{dy'}\right)^2 + 1}}$$

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$$\cos. nx = \frac{-\frac{dz'}{dx'}}{\sqrt{\left(\frac{dz'}{dx'}\right)^2 + \left(\frac{dz'}{dy'}\right)^2 + 1}}$$

Taking as an example, the cone of art. 294, we have

$$\cos. nz = \frac{a}{\sqrt{1 + a^2}}$$

which, since a is the tangent of the half angle, ε, at the vertex, becomes

$$\cos. nz = \sin. \varepsilon,$$

and shows that the system of planes in the second example of art. 276, are the tangent planes of a cone.

SECTION IV.

OF THE SINGULAR POINTS OF CURVES.

Definition and subdivision—of multiple points—criterion by which it is discovered whether a given portion of a curve is concave or convex—points of inflexion—conjugate points—serpentine points—examples.

298. In the theory of tangents and normals, which the differential calculus has enabled us to develope, we have evidently gained an instrument of much use in discussing the forms of curves, but the powers of the same agent may be employed in carrying the investigation of the latter further, and will then enable us to determine their *singular points*, or those points in any given curve that have a character sufficiently peculiar to distinguish them apart from the portions of the curve immediately adjacent.

The points to which we have made allusion, are usually classed under the several heads of: *Multiple points—points of inflexion—points of reflexion—conjugate points* and *serpentine points.*

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Art. 299. Of multiple points.

299. Multiple points are merely the intersections of two or more branches of a curve, and will be found either by taking the equations of the branches simultaneously, or by means of the expressions for the tangents.

Assuming, for example,

$$x^4 - a^2x^2 + y^2 = 0$$

as the equation of a plane curve, we have, by solving it with regard to y,

$$y = +x\sqrt{a^2 - x^2}$$

and

$$y = -x\sqrt{a^2 - x^2}$$

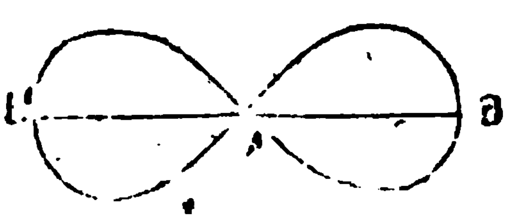
as the equations of the branches.

And regarding these as simultaneous, we deduce

$$x = 0, \text{ and } x = \pm a$$

as values that satisfy them both, and which must therefore belong to points common to both branches. This fact, however, is not sufficient of itself to establish the existence of a multiple point, for as the branches of a closed curve unite themselves into one and the same line, the points found may be those where the branches so unite, and on discussing the curve, fig. 261, we find with regard to two of the values, that such is actually the case; the values $x = +a$ and $x = -a$ indicating merely the points at B and B′, where branches of the curve unite.

Fig. 261.



The third value, $x = 0$, belongs to the point A; and as this is really an intersection of two branches, we see that some criterion is necessary to distinguish, among the values obtained from the preceding process, those

indicating multiple points, from the values that merely assign the place where a branch returns into itself. The mere algebraic discussion of the equation of the curve, when the latter has been solved, is fully equivalent to this purpose; and is indeed the most convenient method that can be used: but as it is not always possible to resolve the equations of curves, mathematicians have had recourse to another method; and have both determined the position of A, and distinguished it from such points as B and B′, by the property that the curve at the point A possesses more than one tangent.

Proceeding on this method of inquiry, we observe, that since the several tangents at the common point will merely be distinguished by their directions, the fact of the point being single or multiple, will depend upon the number of values which the expression $\frac{dy}{dx}$ is found to possess.

Now the equations of the curve containing only two variables, its differential can be put under the form

$$Mdx + Ndy = 0;$$

whence

$$\frac{dy}{dx} = -\frac{M}{N},$$

and denoting either side by α, we have

$$M + N\alpha = 0.$$

But the quantities M and N, when the equation of the curve has been cleared of functions having more than one value, will have, for given values of x and y, but one value each; and as we suppose α to have more than one value, it follows that M is either equal to the same multiple,

of two different numbers, which is impossible, or that $M = 0$ and $N\alpha = 0$.

But, again, since α has, at least, two distinct values, it is not the factor which causes $N\alpha$ to vanish, and we have, finally,

$$M = 0, \quad N = 0.$$

Having arrived at these equations, the process to be used is immediately obvious. For since at a multiple point, the equations, on the hypothesis assumed, are necessarily true; we have merely to regard them as such, and as existing simultaneously with the equation of the curve, in order to determine the co-ordinates of those points where multiple branches *may* exist. It does not follow, however, since the process is founded on a converse proposition, that multiple points *do* really exist there; but if such points are not found in the places so determined, we may then be assured that the curve is without them.

300. A more general principle, that embraces every species of singular points, may be obtained by extending the theory of tangents; a theory, it will be observed, that has with the subject in hand obvious connections, since these lines being directions to which the curve approaches, the form of the curve, at the point of contact, must depend upon the degree of departure from the tangent.

The departure in question is expressed, art. 278, by the equation

$$y_{,} - y = \frac{d^2y'}{dx'^2} \cdot \frac{h^2}{1 \cdot 2} + \frac{d^3y'}{dx'^3} \cdot \frac{h^3}{1 \cdot 2 \cdot 3}, \text{ \&c.} \ldots \alpha$$

and the species of the point sought will, therefore, depend upon the nature of the terms which this series contains.

Chap. II. Relations that exist between the lines or the surfaces of one system, and those of another.

Art. 301. Criterion by which it is ascertained whether a given portion of a curve is concave or convex.

301. With respect to this last we may remark, that, supposing none of the co-efficients to be imaginary, illusory or infinite; and having ascertained which of the terms is the first that does not vanish, h may always be taken sufficiently small to render this first term greater than the sum of all those which remain; a result that renders the series better adapted to the inquiry in hand.

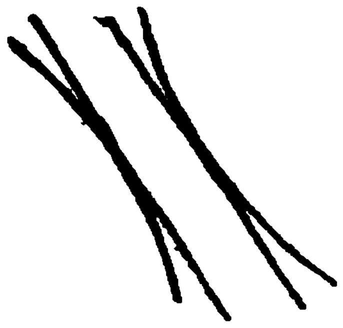
The object of the latter, we recollect, is to discover the form of the curve at the point of contact; and as some of the chief peculiarities in this form depend upon the side on which the curve is concave or convex, it will be useful to obtain a criterion whereby the direction of the concavity can be known.

This criterion is immediately afforded by the series α; for since every curve presents its convex side to the tangent, we have merely to determine the side to which the curve turns; or, in other words, to determine whether $y_, - y$ is positive or negative, in order to obtain the object sought.

Now the sign of $y_, - y$, by what has been said, is determined by the sign of the first term which remains in the series α.

If this term is multiplied into an even power of h, the difference, $y_, - y$, fig. 262, will have the same sign on either side of the point of tangency, and will be convex to the axe of the x's, or concave to it, according as the first term in question is plus or minus.

Fig. 262.



302. Should the first term that remains be multiplied into an odd power of h, the

difference $y_{,} - y$ will change its sign on opposite sides of the point of contact, and the curve on one side will turn its concavity to the axe of the x's, and on the other it will turn its convexity.

Fig. 263.

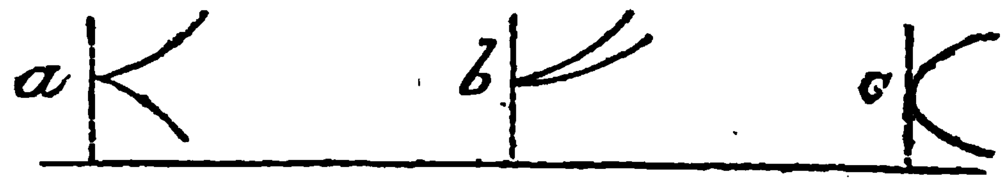
The point of contact is said, in this case, to be a point of *inflexion*. And to discover whether such points exist in a curve, we should commence by regarding the equation of the latter as existing simultaneously with the equation

$$\frac{d^2y'}{dx'^2} = 0;$$

the values of x and y, found from these equations, are then to be substituted in the higher differential co-efficients; and if the order of the first co-efficient that does not vanish, is odd, and the remaining co-efficients are neither imaginary, nor infinite, nor illusory, we may regard the point corresponding to the co-ordinates so found, as a point of reflection.

303. When, for given co-ordinates, any of the terms in the series a become imaginary, not only the difference $y_{,} - y$, but the value of $y_{,}$ itself becomes imaginary; and either the curve must terminate at this point, or turn back towards its former direction.

Fig. 264.



The first of these cases, if not altogether impossible, is at least so with regard to curves represented by any of the expressions of analysis that we have yet seen; for as

in all such expressions imaginary quantities enter by pairs, it will follow that either none, or *two* values of y become imaginary for the same values of x; or, in other words, two branches of the curve must meet and terminate at the same point.

The several cases of this kind that can happen, are shown at a, b and c, fig. 264.

The two first are called points of *reflexion* of the first and second *species*; the third is not regarded as a singular point, since it varies with the directions which the axes of the co-ordinates are made to assume.

A point of reflexion of the first species is distinguished from one of the second by the directions in which the branches turn their concavities; and the same character enables us to distinguish these points from the case represented at c. The branches in the case first alluded to, have their convexities opposed to each other; and hence, in the immediate neighbourhood of this point, the value $\frac{d^2y}{dx^2}$ will have opposite signs in the two branches, the sign for the superior branch being positive, and for the inferior branch negative. The former condition applies also to the case at c; which is distinguished from a point of reflexion, by the positive value of the co-efficient belonging to the inferior branch.

The character of the second species of points of reflexion is that of turning the concavities of either branch towards the same direction, or of giving in each branch the same signs to the differential co-efficient $\frac{d^2y}{dx^2}$; and as this character is not observed in the case delineated at c, either sign may belong to the superior branch.

304. The complete analysis of points of reflexion requires that we should resolve the equation of the curve, in order to determine the separate equations of the branches; and as this resolution may be effected by means of infinite series, and, when accomplished, will lead immediately to a knowledge of the points of reflexion; the assistance of the differential calculus is only required to determine in what directions the convexities of the branches are turned.

A similar remark applies to *conjugate* points; which may, therefore, be determined without employing the series α, but which have with it relations of the same kind as the preceding.

Conjugate points are points detached from a curve, and that stand isolated from every other point belonging to the latter.

They are best illustrated by means of the theory of parameters.

Thus assuming, as an example, the system represented by the equation

$$y^2 - x^3 + (x' + x'')\,x^2 - x'x''x = 0,$$

where x' and x'' are parameters measured on the axe of the x's, but corresponding to different branches of the curve; we have, by solving the equation,

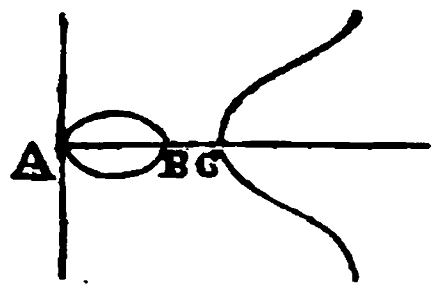
$$y = \pm \sqrt{\{x.\,(x - x').\,(x - x'')\}};$$

where, excluding the cases in which the parameters are either zero or negative, y vanishes with x, and remains positive and real as long as x is positive and less than either of the parameters. Assuming the latter to be arranged in the order of their magnitude, and supposing x to continue increasing, y again becomes zero when x has either of the values $x = x'$, or $x = x''$:

between these values y is imaginary; and beyond the last it continues real, and increasing with x, to infinity.

From the analysis which has been here given, the form of any of the curves of the system, contained within the limits assigned to the parameters, will be such as that delineated in fig. 265, where the points A, B and C correspond to the values $x = o$, $x = x'$ and $x = x''$.

Fig. 265.



Now, observing that $AB = x'$, and supposing the smallest of the two parameters to decrease to zero, the oval AB will in like manner decrease, until, when $x' = o$, it becomes merely a detached point, A, which is then called a *conjugate* point of the curve.

The method of investigating points of this kind when the equation of the curve has been solved, will be apparent from what is here said, since it is merely necessary to examine whether such values can be found for x and y, that if x were made either to increase or decrease by a small indeterminate quantity h, the values of y would be imaginary.

The series α will contain, in this case, imaginary quantities that do not destroy each other when h is either positive or negative.

But as that fact cannot be ascertained without solving the equation, geometricians have had recourse to the property, that conjugate points, as well as points of inflexion, cause the term $\frac{dy}{dx}$ to become $\frac{0}{0}$.

This property may be demonstrated by the process used in relation to points of inflexion; since, arranging, as before, the differential equation of the curve under the form

$$M \frac{dy}{dx} + N = 0 \quad . \quad . \quad . \quad . \quad . \quad 1$$

And continuing to differentiate until we come to that differential co-efficient $\frac{d^n y}{dx^n}$, which is the first to contain imaginary quantities, we have

$$M \frac{d^n y}{dx^n} + P = 0;$$

where P, containing only differentials of an order lower than n, does not become imaginary with the assumed values of x and y.

But as, substituting these values in the equation 2, they cause the first term to become imaginary whilst the second remains real; the equation will reduce itself to the two, $M = 0$, and $P = 0$; the first of which, combined with the equation 1, gives

$$M = 0, \quad N = 0.$$

And agreeing, in this respect, with the results deduced for points of inflexion, supposes an analysis that only differs from the latter, when we come to determine to which of the cases the point discovered is to be assigned.

The process whereby the latter is effected, consists of an algebraic discussion of the values of y at points immediately adjacent to that in question; but as this cannot be accomplished in a general manner without the equation is solved by a series, it is to this we are obliged, in all cases of difficulty, to have recourse.

Prior to leaving the subject of conjugate points, it may not be amiss to notice the fact indicated in that instance by the ambiguous sign $\frac{0}{0}$, assumed by the direction of the tangent. The ambiguity in the case of multiple points referred, we said, to the existence of several tan-

gents, and to the quantity $\frac{dy}{dx}$ having, in consequence, several values.

But in the instance before us, the ambiguity is somewhat different, the symbol $\frac{0}{0}$ signifying that we may give to $\frac{dy}{dx}$ any values at pleasure; a fact, indeed, that might have been suspected, since the oval, before mentioned, having gathered itself into a point, its system of tangents, which corresponded to all possible values of $\frac{dy}{dx}$, have now a common point of tangency.

305. What has been said concerning conjugate points, will render the nature of serpentine and other invisible points more readily understood; for we have merely to suppose such alterations made in the parameters of a curve possessing a conjugate point, as will cause some other branch of the curve to pass through the latter, and we immediately perceive that points of a curve which appear simple, may in fact be complex.

A curve, for example, that has a waving or serpentine form, may be so altered by a change in the parameters, that the wave at A shall diminish in size, still retaining all the characters of its inflexion; until at last it becomes a mere point, no longer to be distinguished, by the senses, from other points in the curve, but still recognized by its analytical characters.

Fig. 266.

Such a point is said to be *serpentine*, and belongs to one of the classes of *invisible* points.

Of the nature of these last, our limits will not permit us to speak further; and we can therefore only briefly notice, that as serpentine points are merely cases of points of inflexion, they require the order of the first differential co-efficient that does not vanish, to be odd; the number of co-efficients which vanish sufficing to show whether the point is of simple or serpentine inflexion.

We shall terminate this subject with some examples taken from LACROIX.

306. 1. Let $y^2 x^2 + a^2 x - a^3 x = 0$ be the equation of the curve.

If $x = \frac{3a}{4}$, y will equal $\pm \frac{a}{\sqrt{3}}$, and there are points of inflexion corresponding with each of these values.

2. Let the equation be $ay^2 - x^3 - bx^2 = 0$.

At the origin of the co-ordinates $\frac{dy}{dx} = \pm \sqrt{\frac{b}{a}}$, and as as this indicates two tangents, there must be a *node* or *double point.*

When $b = 0$, the curve becomes the semicubical parabola; and in this case, the two tangents coincide with each other, and the node is changed into a point of reflexion of the first kind. But if b be negative, the values of $\frac{dy}{dx}$ become imaginary, and this point is an insulated or conjugate point.

SECTION V.

OF CURVES TANGENTIAL AND NORMAL TO SYSTEMS, AND OF THE SINGULAR POINTS OF SYSTEMS.

Curves tangential to a system of curves—curves normal to a system of curves—singular points of a system of curves—surfaces tangential to a system of surfaces—envelope of a system of surfaces—surfaces normal to a system—singular points and lines of a system.

307. The principles of reasoning that led us in articles 277 and 278 to determine a system of straight lines tangential to a given curve, would have sufficed where the touching lines were themselves curves; and thus, reversing the problem, we may either seek a system of curves that shall touch a given curve, or the curve that shall touch a given system.

The analytical conditions when the osculating lines are both curved, is still the same as in art. 278—the lines will have the same tangent at the point where they meet, and will therefore require the first differential

co-efficients to be equal. But although the same analytical condition is common to both problems, they cannot be solved by following analogous routes; since in the latter problem, the species of the curve sought is not given; a restriction that was observed in regard to the former.

Assuming, on this account, the curve sought to be represented by the general equation

$$\phi\,(x, y) = 0,$$

and the system of given curves by the known equation

$$\mathrm{F}\,(x', y', p) = 0;$$

we must seek from these equations to fulfil the condition

$$\frac{dy'}{dx'} = \frac{dy}{dx}.$$

Now differentiating the given equation; and eliminating, by means of the primitive, p from the result; we observe that the equation so deduced, and which we will denote by a, must exist simultaneously with the result obtained by differentiating the equation

$$\phi\,(x, y) = 0,$$

since the values, of $x,'$ y' and $\frac{dy'}{dx'}$ that satisfy one, are, at the points of contact, the same with the values of $x,$ y and $\frac{dy}{dx}$ that satisfy the other.

This analysis immediately reduces the inquiry to a problem of the differential calculus; since, as we have formed a differential equation, a, of the curve sought, the integral of that equation must be the curve itself.

This integral, however, was limited not to contain p, since otherwise the touching curve would vary with the curve that was touched; and hence it must be either

one of the integrals obtained by substituting particular values of p in the equation

$$F(x, y, p) = 0$$

or it must be an integral not contained in the latter.

Now the equation

$$F(x, y, p) = 0$$

is, itself, an integral of a; and containing an arbitrary constant p that a does not contain, will, according to the known rules of analysis, include every integral of the equation, excepting those known as *singular solutions*; and which last, not containing the parameter p, and not being included in the system expressed by the equation

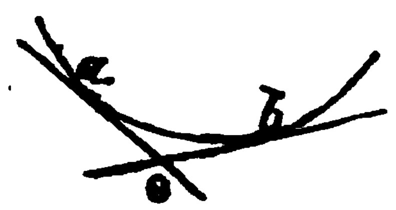
$$F(x, y, p) = 0$$

represent curves such as we require.

But although this solution becomes important from the relations that it bears with the integral calculus, it leads immediately to another, better adapted to the purpose under discussion, and which we shall now explain.

Assuming, with that view, any two consecutive points a and b, in the curve sought, we may regard them as points of tangency to two consecutive curves of the system.

Fig. 267.



And as within such distances from the point of contact as are expressed by the small, but variable quantity dx, the difference between the curve and tangent involves no quantities of an order lower than d^2x; we may substitute, whilst speaking of first differentials, the tangent in place of the curve.

Now the tangents at a and b will meet, when produced, at some point O; and as the difference between O and a point in the curve is expressed by a second differential,

we are, again, permitted to neglect such quantities, and to regard, whilst reasoning on first differentials, the curve that passes through all the points O, as identical with the curve that passes through the several points of tangency.

Or, since the points analogous to O are formed by the intersections of consecutive tangents, we may determine the curve which is the object of the present investigation, by seeking the intersections of each pair of consecutive tangents. The equation of the curve, when so determined, will, it is true, involve the parameter p; but as this last may be eliminated by the assistance of the equation of the system, we shall, finally, arrive at an equation of the tangential curve, involving no other variables but x and y.

The method to be pursued in determining the intersection in question, will be immediately apparent; since it will be merely necessary to obtain the equations of two consecutive lines in the system, and regard them as existing simultaneously.

The equation of any one of the lines in question being obtained by assigning the value of p, in the equation

$$\mathrm{F}(x, y, p) = 0, \quad \ldots\ldots b$$

that of the consecutive line will, in like manner, be obtained by giving the same value to p, in the expression

$$\mathrm{F}(x, y, p + dp) = 0; \quad \ldots\ldots c$$

whence, expanding c, subtracting b from the result, and neglecting the differentials of an order higher than the first, we obtain

$$\frac{d \,.\, \mathrm{F}(x, y, p)}{dp} = 0, \quad \ldots\ldots d$$

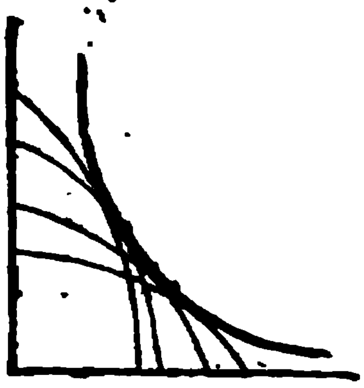
which, taken simultaneously with

$$F(x, y, p) = 0,$$

is the equation of the curve sought.

Let there be given as an example illustrating this theory, the system of parabolas represented by the equation

Fig. 268.



$$y + y^{02}x^2 - y^0 = 0,$$

where y^0 is the value of y corresponding to x^0.

Differentiating, with regard to y^0, we obtain, for the equation d,

$$2y^0x^2 - 1 = 0;$$

and combining this with the equation of the system, and eliminating y^0, there results

$$4x^2y = 1,$$

as the equation of the tangential curve.

Taking, as a second example, the equation

$$y^2 + y^{04}x^2 = y^{02},$$

that is shown in article 258 to belong to a system of concentric ellipses, we have, by differentiating with regard to y^0,

$$2y^{03}x^2 - y^0 = 0,$$

an equation that is equivalent to the two

$$y^0 = 0$$
$$2y^{02}x^2 - 1 = 0$$

taken separately.

Combining the second with the equation

$$y^2 + y^{04}x^2 = y^{02},$$

we deduce

$$4x^2y^2 - 1 = 0$$

as the equation of the curve tangential to the system; and as we may first divide this last result into the two equations

$$2\,xy - 1 = 0$$

and

$$2\,xy + 1 = 0$$

we discover the curve sought to be formed of two hyperbolas, having the axes as their asymptotes.

It will be observed that we have not yet employed the solution $y^{\circ} = 0$, that equally resulted from the preceding analysis; this value substituted in the equation of the curve leads to

$$y = 0$$

or to the axe of the x's; a line that is not tangential to any of the curves in the system.

This apparent anomaly will be explained by referring to the method of analysis employed, and which merely requiring the value of $\frac{dy}{dx}$, corresponding to given values of x and y, to be the same in the system and the curve, leads to every solution that possesses this property. Now, in the case before us, the curves corresponding to an indefinitely small value of y°, differing from lines parallel to the axe of the x's, by an indefinitely small difference, have every where the same value of $\frac{dy}{dx}$.

308. The lines normal to the system will be obtained by a process similar to the first of those used in obtaining the curves that were tangential, and which may indeed be

regarded as a general method of finding curves that shall form any given angles with a system of known curves.

Representing, as before, the system by the equation

$$F(x', y', p) = 0;$$

taking the differential

$$\frac{d\,F(x', y', p)}{dx'} = 0; \quad . \; . \; . \; . \; . \; e$$

and observing that in the normal curve, we have

$$x = x', \quad y = y', \quad -\frac{1}{\frac{dy}{dx}} = \frac{dy}{dx},$$

the curve in question will be found by substituting these values in the equation e, and, by means of the equation of the system, eliminating the parameter.

Assuming, for example, the system of similar and concentric ellipses, expressed by the equation

$$y^2 + a x^2 = y^{\circ 2},$$

we obtain, by differentiation,

$$y\frac{dy}{dx} + a x = 0;$$

and, making the substitutions in question, there arises

$$y - a x \frac{dy}{dx} = 0;$$

which, not containing the parameter y°, is the equation of the system sought.

Separating the variables and integrating, we deduce

$$y^a = cx$$

for the equation of the system of normal curves; the arbitrary constant a being in this case the parameter.

Sect. V. Of curves tangential and normal to systems, and of the singular points of systems.

Art. 309. Singular points of a system of curves.

309. Reasons analogous to those used in relation to the singular points of curves will lead us to perceive that systems have also their *singular lines* and *points*, and *lines of singular points*, which are determined by a similar analysis: but as this subject has not been investigated, it would be improper, in an elementary treatise, to do more than indicate the existence of such relations.

To illustrate the subject, let us assume the system represented by the equation

$$y = p + (x - p)^3.$$

Taking the second differential, and equating it with zero, we have

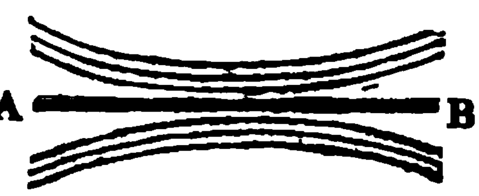
$$x = p$$
$$y = p;$$

whence, since the curves taken separately have each a point of inflexion, a fact that will appear from art. 302, these points must all fall in a straight line passing through the origin, and making an angle of 45° with the axe of the x's. Such a line we have termed a *line of singular points*, intending to include in the term, lines made up of the singular points found in the individual curves of the system.

A *singular* line will differ from the last, in being itself a part of the system; namely, some one line belonging to the latter, that is distinguished by a character not found in the remainder to the same extent.

The straight line AB, for example, that serves as a boundary between the lines in fig. 269 that are concave, and those which are convex, is "a singular line" of this system: and a similar remark may be extended to all lines

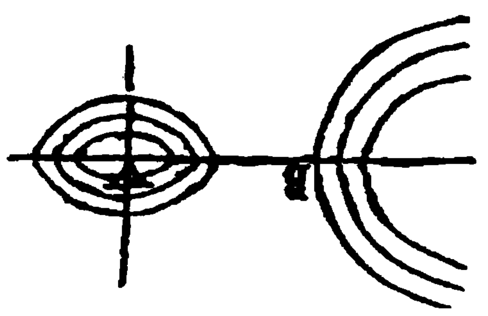
Fig. 269.



that, being parts of the system, are, also, boundaries; as to the inner and outer boundaries of the system delineated in fig. 254.

And, in the same way, a *singular point* in a system is a point in the neighbourhood of which the curves approach a form that is lost, or departed from more widely as we leave that point. An example of this kind may be taken from the systems of curves observed in the different species of grained woods, where the knots are singular points, round which the curves assume a character not observed in other parts of the system.

Fig. 270.



The point A, fig. 270, is a singular point of the system, expressed by the equation

$$y = \pm\sqrt{\{(x^2 - p^2)(x - (a - p))\}}:$$

being a centre round which the *nodes* of the system are formed.

310. The theory of tangential curves, that occupied our attention in the preceding section, may be used to show the existence of a similar relation among surfaces. For, supposing a system of curves, such as that in fig. 268, to revolve about one of the axes as a diameter; each curve will generate a surface that osculates with the surface generated by the tangential line: and as this is true of every curve in the system, the revolution of the whole will produce a system of surfaces, each touched throughout the circumference of a circle by the tangential surface in question.

Nor is it necessary, except for the purpose of illustra-

tion, to suppose the system of surfaces to be thus formed by revolution; the same principles that were our guide in regard to curves, will equally apply to a system of surfaces.

Thus arguing as in art. 307, that when first differentials only are used, the line wherein two surfaces intersect may be regarded as a line in the surfaee of tangency; we shall investigate the latter by mingling the equations of two simultaneous surfaces, and finally, eliminating the parameters from the result.

Let us assume for example, the system of ellipsoids

$$x^2z^{02} + y^2z^{04} + z^2 = z^{02}$$

mentioned in article 276.

Differentiating with regard to z^0, we obtain

$$x^2z^0 + 2\,y^2z^{03} = z^0;$$

which is equivalent to the two equations

$$z^0 = 0$$

and

$$x^2 + 2\,y^2z^{02} = 1.$$

Employing the latter we derive

$$z^{02} = \frac{1 - x^2}{2\,y^2};$$

and substituting this value in the equation of the surface, and simplifying the result, there arises

$$2\,y\,z = 1 + x^2;$$

where, as the sections parallel to two of the co-ordinate planes, are parabolas; and the sections parallel to the third plane, hyperbolas; the surface is the hyperbolic paraboloid of art. 252.

When the system of surfaces contains two independent

parameters, art. 275, the results will become more general. Since regarding one of them to remain constant, whilst the other varies, there will be a distinct tangential surface for every value given to the parameter which is regarded as constant: and a similar result will follow, if we assume one of the parameters to be a function of the other; since, in this case, we shall have a distinct tangential surface for every form assigned to the function in question.

Systems with two independent parameters *will thus* have an infinity of tangential surfaces; but in each, it may be remarked, the curve of contact is of the same degree.

This fact is readily established; for, supposing the equation of the surface to be

$$F\ (x, y, z, \alpha, \beta) = 0,$$

and putting it under the form

$$F\ (x, y, z, \alpha, \phi\alpha) = 0,$$

where one of the parameters is assumed to be an arbitrary function of the other, we may reason as follows:*

Differentiating this function, with the view of deriving the intersection of any two consecutive surfaces, we obtain, for the equations of this intersection,

$$F\ (x, y, z, \alpha, \phi\alpha) = 0,$$
$$\frac{dF}{d\alpha} + \frac{dF}{d\phi}\phi'\alpha = 0.$$

But since ϕ relates only to the connection between the variable parameters α and β, and does not affect the manner in which they are attached to the variables x, y

* The remainder of the proof is abridged from Le Roy's An. Geo. p. 211.

and z, it will follow that, whilst α and β remain the same, the two preceding equations will always involve x, y and z to the same degree. The intersection is, on this account, called the *characteristic*.

311. The tangential surfaces that we have considered, are called by MONGE the *envelopes* of the system; as they correspond, however, to the tangential lines of plane curves, I have merely spoken of them by their property of tangency; and one of these tangential surfaces being distinguished from the rest, by the fact that it envelopes the whole compound system, I have, for want of any other method of distinction, used the name "envelope" as implying this particular tangential surface.

Its equation is obtained by differentiating the equation

$$F\ (x, y, z, \alpha, \beta) = 0,$$

with regard to each of the independent parameters α and β, successively, and equating the results with zero.

The equation

$$\frac{d\,F}{d\,\alpha} = 0,$$

obtained by the first of these differentiations, expresses, when taken in conjunction with

$$F = 0,$$

the intersection of two consecutive surfaces in that simple system, for which α alone varies; whilst the equation

$$\frac{d\,F}{d\,\beta} = 0$$

has a similar signification in regard to the system where β varies.

When, therefore, the three equations

$$F = 0$$

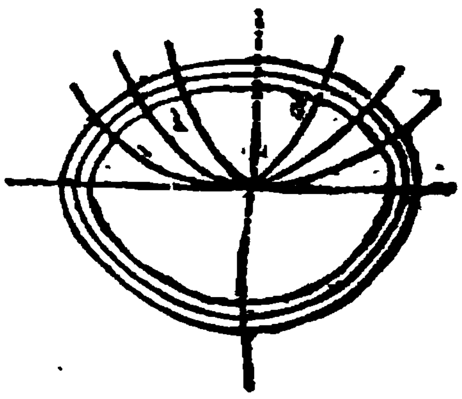
$$\frac{dF}{d\alpha} = 0$$

$$\frac{dF}{d\beta} = 0$$

are regarded as simultaneous, the result belongs to the point wherein the two preceding curves mutually intersect. And, eliminating the parameters α and β, whereby these points are distinguished, the result will be a surface common to them all—the envelope, namely, of which we were in search.

312. After what has been here said, the theory of surfaces that are normal to a system will only require a passing notice. Their existence may be illustrated, as in the preceding case, by the revolution of a system of plane curves upon one of the axes. Assuming for example the system of concentric ellipses, art. 308, to revolve, with their normal curves, about the axe of the z's, and the form of the resulting systems of ellipsoids and normal surfaces will be understood from fig. 271.

Fig. 271.



The analytical process to be employed in cases where the given surfaces are not of revolution, is merely an extension of that used in art. 308, which teaches us, first, to substitute in the differential of the given equation x, y and z, for x', y' and z', and secondly,

after replacing $-\frac{1}{\frac{dz}{dx}}$, and $-\frac{1}{\frac{dz}{dy}}$, by $\frac{dz'}{dx'}$ and $\frac{dz'}{dy'}$, to integrate the result.

313. In concluding the subject, we may remark that systems of surfaces will also admit the relations of singular points and lines noticed in art. 309; as well as some that are peculiar to themselves; such as the line formed by the intersections of the consecutive characteristics, or, in other words, the curve tangential to that system; it forms a species of ridge known by the name of the "edge of regression," and in developable surfaces is always a right line.

NOTES.

Note 1, page 124.

The hesitation of mathematicians to introduce the notion of infinity into treatises of elementary geometry, seems contrary to every principle of sound logic. The idea is an essential element in the relations of place and position, and can only be hidden from view by having recourse to an ambiguity of expression, that renders the demonstrations where it is used, more specious than just. The very idea of an angle involves the infinity of space: since we cannot define an angle to be the opening formed by two lines that meet in a point, without admitting the dilemma, that either this opening is a space enclosed on all sides, which cannot be allowed; or that it is a space extending in one direction to infinity. The usual method of avoiding the latter alternative is, to allow the student to discover for himself the idea attached to the word *opening;* an artifice, that removes nothing of the real difficulty, but has the effect of giving to the subject an appearance of rigour, foreign to its nature.

Expecting that many will object to my views on this point, I have looked over every theory of parallel lines that fell in my way; from that of Euclid, to those recently published in the Southern Review, and in an American edition of Legendre, and can discover none, without excepting even Legendre's celebrated functional demonstration, that are not altogether based on the idea of infinity.

With regard to the functional demonstration, I wish to say a few words:

The two principles whereon it is founded, are, first, that all geometrical relations, whether of number, quantity or position,

can be expressed by equations. And, secondly, that angles and lines are necessarily heterogeneous, *because* the former are numbers, which the latter are not.

As an axiom immediately to follow the doctrine of pure number, and by which the student is to discover the nature of quantity and position, and their relations, the first of these propositions is manifestly inadmissible. It is, in fact, only known to be true, when we have learned that all the relations of quantity, and space, and position, reduce themselves to mere relations of number; a fact far from being at first obvious, and that cannot be demonstrated without using the principle which Legendre employs this axiom to establish.

With respect to the second proposition, that angles and lines are heterogeneous, because the former are numbers; it may be remarked, first, that angles and lines are only heterogeneous from involving, when compared with the whole of space, a different power of infinity; and not in the sense, that a physical quantity is heterogeneous to one purely geometrical. And, secondly, that *neither* angles nor lines *are numbers*, but that both can be *expressed* by numbers, when referred to their several units, namely, the whole of plane space, and an infinite straight line. The true difference between them, lies in the nature of the factor whereby they are referred to those units; and this difference cannot be properly exhibited, without stating, that in one case the factor is infinite.*

It remains for me to mention, that in the demonstration to which this note is attached, I took the idea of expressing the angles of the triangle in terms of its area, from a demonstration suggested by my predecessor at the university.

Note 2, page 125.

In place of demonstrating in each particular case of the problem the dependence of commensurate and incommen-

* It may perhaps be worthy of notice, as a curious fact, that a theory which has occasioned so much discussion when applied to lines described on a *plane*, presents no difficulty when applied to lines described on a *sphere*.

surate quantities, it was thought advisable to defer a truth of such universal application, until it could be established with complete generality.

The student, in this view of the subject, is made, first, to investigate the relations of commensurate quantity, and, subsequently, to extend them, by a single operation, to those which are incommensurate.

This step, which it was intended to make by a process of algebra, I have since replaced by the following geometrical demonstration, that ought to have found a place in the body of the work.

1. The demonstrations of all geometrical problems have been shown to proceed by the continued superposition of elementary figures.

2. When this superposition is made with regard to parts that are incommensurable, one of the points wherein an angle of the last elementary figure ought to fall, is not attained.

3. The error may be diminished to less than any assignable quantity.

4. The directions taken by the sides of the elementary figures, and whereby the points, not determined by superposition, are found, are the same in one case as the other.

5. The only difference, then, which occurs between the analysis of incommensurate and that of commensurate quantities, may be stated, by saying, that, a definite number of the points whereon the figures depend, and, consequently, the figures themselves, differ by less than any assignable quantity.

6. But to prove that figures have no assignable difference, is merely to establish their agreement by an indirect process.*

Note 3, page 191.

Analysing the triangle fig. 113, p. 131, into two right angled triangles, we have, art. 80—2,

* It is remarkable that Mr J. Young, whose works have reached America since this book was in the press, should blame Legendre for his demonstrations of incommensurate quantities, the happiest invention in a masterly work.

$$\text{Area A B } m = \tfrac{1}{2} \text{ A } m \times \text{B } m$$
$$\text{Area B C } m = \tfrac{1}{2} \text{ C } m \times \text{B } m$$

Hence, by addition,

$$\begin{aligned} \text{Area A B C} &= \tfrac{1}{2}\, a' \times \text{B } m \\ &= \tfrac{1}{2}\, a' \times a \cdot \frac{\text{B } m}{a} \\ &= \tfrac{1}{2}\, a' \times \sin.\, aa' \end{aligned}$$

which is the expression used in the text.

Note 4, page 204.

In art. 212, it is shewn that six of the *relations of four points* must be given to determine the remainder; now in the type of closed solids, it is shown to follow from the construction, that three of the angles are right angles, and consequently we have three parts yet to be given before the remainder can be formed: one of the remaining parts, the fourth right angle, is shown in art. 61 to be a necessary consequence of the three other right angles, and is therefore not assumed as a fourth given part.

Note 5, page 242.

Algebra consists of two parts, the method of putting problems into equations, and of interpreting the results when obtained; these last, if the reasoning has been correct, must always give a correct account of the errors made in the premises, and consequently, of the changes required to correct them.

In the case where the errors are those of addition and subtraction only, or of the positive and negative sign, the method of interpreting the results is explained in all the elementary treatises upon algebra.

Note 6, page 295, written by *an inadvertency* note *b*.

The introduction of this notation, and of the very useful case of it where the exponent is negative, is claimed by Babbage: it is however so obvious as to have been in frequent use before the works of that eminent mathematician were in general circulation.

Note 7, page 334.

See art. 204. It was intended to add here some further remarks to those in the text, but as the second demonstration in art. 145, is only intended for students that have made some progress, they appear unnecessary.

Note 8, page 393.

It has escaped the numerous editors of Legendre, that his demonstration of this problem is erroneous. It may, however, be rendered complete by a process of which it is only necessary to mention the general principle.

Assuming A and B for the points on the sphere, *m* for the point taken in the supposed shortest line, and B*n* for an arc of AB, equal to the arc B*m*; describe with a radius A*n*, a small circle which has A for its pole.

1. The shortest distance from A to any part of this circle will always remain the same.

2. The point *n* lies in the circumference of the circle, and the point *m* without.

3. Hence the distance from A to *m* is always greater than the distance from A to *n*; which is the step in Legendre's reasoning that is established on erroneous principles.

THE END.

ERRATA.

Page 5, bottom line, *for* on, *read* or.

Page 17; one of the letters B, in this *figure*, should have *two accents*.

Page 39, line four from bottom; *after* A′B′C′ insert *and* ABC.

Page 109, table No. 7, last line, *for* a''' *read* a^{v}.

Page 120, fourth *and* ninth line from bottom, *for* fig. 103 *read* fig. 104.

Page 121, the *Art.* 48 is *repeated.*

Page 122, the *number* of the *first figure* in this page, is the same with that of the *last figure* in *page* 120.

Page 126; in the *figure*, and that part of the *text* which refers to it, c is put *for* a.

Page 127, ninth line from bottom, *for* convictions *read* conventions.

Page 141; the letter B, at the *bottom* of the *figure*, should be accented; and the *last letter*, in the same page, should be $B_{,}$. Throughout the *equations* in the same page, a'' *should be* a''', and b' *should be* b''.

Page 142, fifth line from bottom, *for* $(b'a'')$ *read* $(b''a''')$.

Page 143; in the *figure*, *for* c *read* e.

Page 146; throughout this page, *for* c' *read* $c_{\prime}$.

Page 147; do. do.

Page 152, fifth and seventeenth line from top, *for* A′ *read* $A_{,}'$.

Page 154; in the *figure*, the letter E, which occurs in the line AE, *should be* F.

Page 155, ninth line from bottom, *for* $\frac{1}{2}$ (AA″) + (A″A‴) + (AA⁗), *read*, $\frac{1}{2}\left\{(AA'') + (A''A''') + (AA'''')\right\}$.

Page 170; in the *figure*, the letters A and E are *omitted.*

Page 219; the letter x, adjacent to P′, in this *figure*, *should be* z.

Page 222; the *number* of the *figure* 172, is *omitted.*

Page 263; the combination 3, is merely the preceding combination written backward; it *should be* $a \quad b' \quad \bar{c} \quad \bar{a}'$.

Page 278, sixteenth line from top, *for* Art. 120 *read* Art. 119.

Page 280, seventh line from top, *for* 6·2883185 *read* 6·283185.

Page 290, eleventh line from top, *for* articles 19 and 20 *read* articles 119 and 120.

Page 292, formula 17, *for* $\frac{\sin. a - \sin. \beta}{\cos. a + \cos. \beta}$ *read* $\frac{\sin. a - \sin. \beta}{\sin. a + \sin. \beta}$.

Page 311, eighth line from top, *for* art. , *read* art. 49.

Page 311, tenth line from top, *for* art. , *read* art. 129.

Page 328, fifth line from bottom, *for* supposing BCD situated at its angle, *read*, supposing ABCD situated at its angles.—The angles wholly acute are situated at A and B.

Pages $\left.\begin{matrix}369\\372\end{matrix}\right\}$ two figures 208.